STRUCTURAL DESIGN

OF

METAL AIRPLANES

F. A CRANE

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SECTION I

THE DESIGN PROBLEM

PREFACE

Progress in airplane design in this country during the last few years has been phenomenal. This progress has been exclusively in all-metal construction. Of the types of metal construction, the trend is toward the general use of thin sheets formed into structural stress members. At the time of this writing there is no text on the market purporting to present the fundamental principles and methods involved in this type of metal construction. It appears, therefore, that the time is favorable for a text for college students presenting the basic principles and methods of metal-airplane design. This text is the author's attempt to fulfill the need.

The manuscript as originally prepared would have required a volume approximately three times as large and expensive as this text. After a study of the material, in collaboration with other aeronautical engineers and teachers of aeronautical structural design, the material was completely revised and the manuscript rewritten, retaining the essence of the original material, but reducing the volume so that its size and content would make it applicable to general use by college students in airplane-design classes.

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UNIVERSITY OF CALIFORNIA,
April, 1935.

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list these in order of importance, since opinions differ as to the most important considerations. Similar lists may be prepared for other types of airplanes.

SPECIAL REQUIREMENTS FOR THE DESIGN OF TRANSPORT AIRPLANES

1. *Minimum first cost*—requiring a minimum of capital invested.

2. *Small size*—requiring a minimum amount of housing facilities. This again reduces the capital invested.

3. *Simplicity of construction and assembly*—to permit quick and adequate servicing.

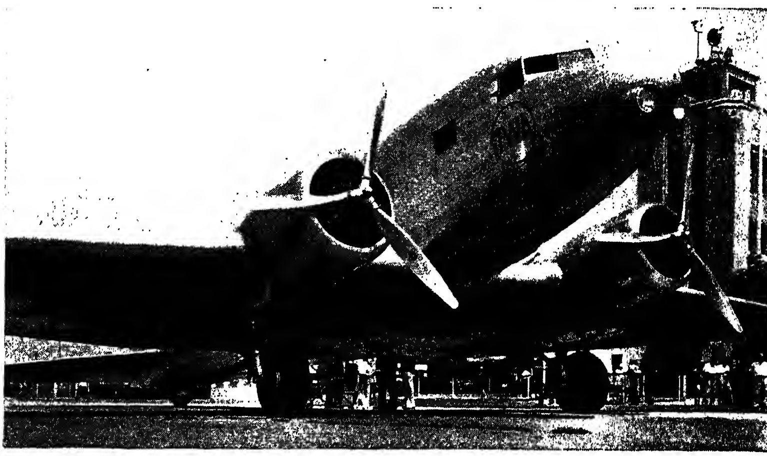

FIG. 3.—Douglas Air Liner; all-metal stressed skin construction. (*Courtesy of Transcontinental and Western Air.*)

4. *Rugged structure and power plants*—requiring a minimum of repair and maintenance and permitting a maximum number of hours per day in the air.

5. *Maximum structural strength*—to sustain the most severe aerodynamical load which may be imposed in the air or in landing or take-off conditions in bad fields.

6. *Minimum gross weight*—for maximum performance in climb and ceiling, etc.

7. *Minimum parasite resistance*—for a maximum cruising speed.

8. *Reasonable wing loading*—for a not too excessive landing speed and good performance and handling qualities at the altitudes at which the airplane may be operated.

9. *Maximum of power available*—for the allowable gross weight; also for best climb and take-off speed.

10. *Maximum of comfort for passengers.*

11. *Maximum of comfort for pilots*—good vision from the cockpit and logical arrangement of controls and instruments.

CHAPTER I

DESIGN REQUIREMENTS

1. Structural and Aerodynamic Efficiency.—In airplane design, we should constantly bear in mind that any airplane represents a compromise between aerodynamical and structural efficiency. For example, a monoplane with a very thin wing section might be, for certain flight conditions, an ideal type of airplane, yet it

FIG. 1.—Martin Army Model XB-10. High-speed bomber. Pioneering design in all-metal stressed skin construction in the United States. (*Courtesy of The Glenn L. Martin Company.*)

would be structurally impractical to build for the weight would be prohibitive. While the braced-wing airplane is probably the most efficient structurally, it is inefficient aerodynamically because of the drag of the bracing which is exposed to the air stream. Structurally the efficiency ratio may be expressed as the *ratio of strength to weight.* Aerodynamically, the efficiency ratio is expressed as the *ratio of lift to drag.*

2. General Specifications.—Design elements which the engineer must consider may be listed as follows:

I. Loads to Be Carried

1. *Crew.*—Pilot, assistant pilot, radio operator, mechanics, and attendants.
2. *Fuel and oil.*—Depend upon range of operation.
3. *Equipment.*—Instruments, radio, lavatories, seats, and other accommodations.
4. *Pay load.*—Passengers, mail, baggage, and express.

II. Performance Desired

1. *Cruising speed.*—This item depends upon the purpose of the airplane. Economy may be the deciding factor in some cases; in other cases, such as for high-speed passenger transportation, high speed is the criterion.
2. *Landing speed.*—At present, the maximum landing speed allowable is prescribed for all commercial airplanes by the Aeronautics Branch of the Department of Commerce. It is desirable from the standpoint of safety to have the landing speed as low as possible.
3. *Rate of climb.*—Enhances safety after take-off.
4. *Service ceiling.*—Important in clearing mountain ranges and low weather disturbances.
5. *Range.*—In normal transport operation, the economical range is 400 to 600 miles.

III. Power Plant Available

1. *Power* to be used.
2. *Number* of power units.
3. *Type of engine.*—Air cooled, water cooled, or chemically cooled.
4. *Geared* or *direct* drive.
5. *Supercharged* or *nonsupercharged.*

IV. Type of Propeller

1. *Material.*—Wood, fiber, duralumin, or steel.
2. *Arrangement.*—Two-bladed, three-bladed, or four-bladed; controllable pitch.

V. Structural Arrangement

1. *Biplane or monoplane.*—There is quite a variation of opinion as to the merits of each of these. The student is advised to refer to the bibliography for detailed study of the question.
2. *Disposition of tail surfaces.*—Depends upon general structure—whether monoplane, biplane, etc. For single-engined ships, the monoplane tail is generally used. For multi-engined ships, two or three fins and rudders are often necessary.
3. *Landing gear arrangement.*—Retractable or otherwise.
4. *Fuselage.*—Shape and arrangement.
5. *Cabin.*—Interior arrangement.

VI. Factor of Safety and Degree of Reliability

Government requirements insure certain safety precautions. The designer should, however, especially in the design of transport airplanes, consider safety as the first prerequisite in design.

VII. Aerodynamic Considerations

1. *Fineness ratio.*—Performance characteristics are improved by making a "clean" design. This consists in eliminating protuberances as much as possible and forming or streamlining those which must be outside.

2. *Stability.*—Probably, in time, no airplane will be considered satisfactory that is not positively stable laterally, directionally, and longitudinally under all flight conditions.

3. *Controllability.*—Implies the proper design of controls and control surfaces.

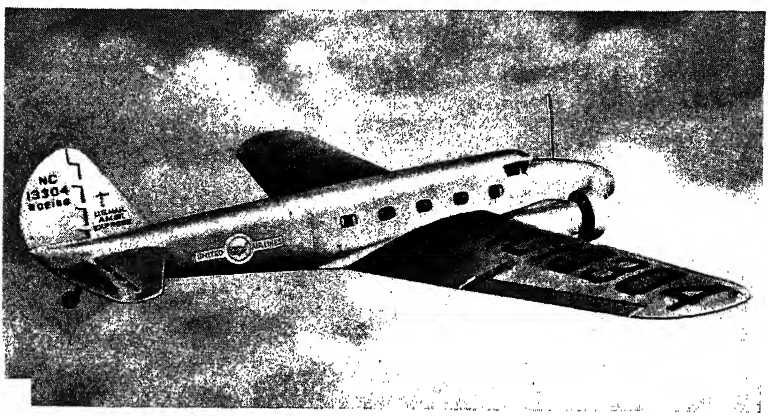

Fig. 2.—Boeing Transport, (247), all-metal stressed skin fuselage; two spar metal covered wings. (*Courtesy of United Air Lines.*)

VIII. Weight

Weight saving is said to be the principal worry of aeronautical engineers. This implies careful and accurate structural design.

IX. Materials

The principal materials now used are wood, steel, and duralumin (see Sec. II).

3. Special Requirements.—For each purpose of an airplane there is a specific set of requirements for its design. As an example of such requirements, consider those for a transport airplane. A brief summary of the major specifications presents its own story of the complex nature of the problem. It is impossible to

It is apparent, therefore, that the successful design of large airplanes requires that the wing loading be increased.

To some extent the *cube law*, that is, the law of the increase of weight in proportion to the cube of the proportionate increase in size, may be defeated by careful detail design. However, there is a limit to this possibility, so that further increase in size requires an increase of wing loading. This increase of wing loading, of course, requires an increase of lånding speed, unless some means of increasing $K_{y(\text{max.})}$ is used, such as, for example *wing flaps*. There is a limit even to this possibility, so that a further increase in size may be said to require an increase in landing speed.

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12. *Adequate and easily accessible cargo space*—for pay load other than passengers.
13. *Adequate gas capacity*—sufficient to make longest probable trip against a probable prevailing head wind.
14. *Installation of best available aids to navigation*—including radio.

4. Ultimate-strength Requirement.—The ultimate strength of materials is the basis of structural-strength analyses in this country. This requirement is the outgrowth of early attempts to produce theoretically the breaking-strength loads which were obtained by static testing of airplanes. This requirement imposes a heavy burden on the stress analyst because no theoretical process of analysis has been devised for calculating stresses above the proportional limit of the material. The mathematical theory of elasticity, in which field we find our engineering formulas, is based on Hooke's law of the *proportionality of stress and strain,* and hence does not apply above the proportional limit of the material. The student is warned that the value of the modulus of elasticity is constant only below the proportional limit of the material.

5. Flexibility, Rigidity, and Strength.—Except for the last three or four years, practically no consideration has been given, in government requirements, to the flexibility and rigidity of the component parts of aircraft. However, as expressed by one of our prominent engineers recently, "The problem of designing a structure so that there will be sufficient rigidity to resist all vibratory or fluttering influences is a problem which is really more difficult than the strength problem."

There is a great deal of difference between *rigidity* and *strength*. A structure may be very rigid, yet very weak. It may be very strong, yet very flexible. For example, a tube of bond paper is very rigid in torsion. However, it is very weak in strength. On the other hand, an India-rubber rod is fairly strong but very flexible. We note that it is possible, in designing a landing-gear axle for the ultimate strength only, to make it entirely unsatisfactory for use because of its flexibility. For instance, assume the ultimate shear strength to be 150,000 lb. per square inch. Assume that the axle has considerable length. In actual test, to develop this stress of 150,000 lb. per square inch, it may be necessary to twist the axle through one revolution or more. It is quite evident that all the fittings and other accessories on this axle would have been sheared off before this ultimate strength

is found. Numerous engine-mount designs have proved sufficiently strong but quite flexible—so flexible, in fact, that the operations of the engines were very unsatisfactory. Flexibility, in this particular case, may be desirable, yet we should recognize the fact that the question of flexibility does occur and is a very important consideration.

The design of the mountings of engines so that they will have the proper degree of flexibility illustrates the importance of this problem. If the engine is built rigidly into the nacelle or into the fuselage, quite naturally, then, all the vibratory stress will be transmitted directly to the airplane structure. On the other hand, if the engine is attached to the fuselage through the medium of springs or rubber bearings, the engine will oscillate as a unit and the stress will not be transmitted directly to the airplane structure. In the first case, that of the rigid structure, the stress in the structure which is required to hold the engine will be excessive. On the other hand, in the case of the flexibly mounted engine, the vibrations on the engine will be so great that parts of the engine may actually be shaken off. Cases have been known where the magneto and the carburetor and other accessories have been thrown off. In designing the supports for the engine, this question has to be decided, and we must strike a happy medium between the problems of *strength*, *flexibility*, and *rigidity*.

6. Weight as Affected by Size.—The weight of geometrically similar bodies varies as the cube of corresponding dimensions. For example, if a cubical block is doubled in size, that is, each edge doubled in length, its volume and weight will be eight times the volume and weight of the original block. Likewise an airplane doubled in size will be eight times as heavy, assuming, of course, that geometrical similarity is maintained. An airplane doubled in size, however, has only four times the wing area. Assume, for example, a 1,400-lb. airplane with an effective wing area of 100 sq. ft. and a wing loading of 14 lb. per square foot. If the airplane is doubled in size, its weight will be 11,200 lb. and its effective wing area will be 400 sq. ft. The wing loading will therefore be 28 lb. per square foot.

Since a wing loading of 14 lb. per square foot corresponds to a landing speed of 60 to 65 miles per hour, the landing speed in the second instance will need to be higher than the 65 miles per hour maximum allowed by law in this country. The landing speeds,

or stalling speeds, as may be noted in books on aerodynamics, are determined from the lift formula (engineering units):

$$K_y A V^2 \tag{1}$$

in which L is the lift in pounds, A is the area of the wings in square feet, and V is the velocity in miles per hour. Taking L as the

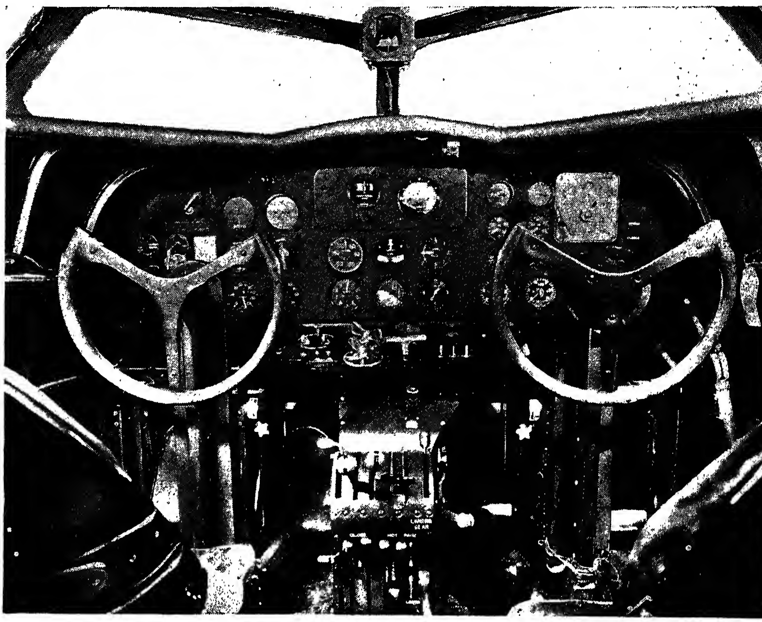


Fig. 4.—Pilot's compartment, Boeing Transport (247). (*Courtesy of United Air Lines.*)

gross weight W of the airplane, and $K_{y(\text{max.})}$ as the maximum lift coefficient, the landing speed is

$$V \quad \sqrt{\frac{W}{K_{y(\text{max.})}A}} \tag{2}$$

Since W/A is the wing loading w, therefore

$$V = \sqrt{\frac{w}{K_{y(\text{max.})}}} \tag{3}$$

Now the average magnitude of $K_{y(\text{max.})}$ for high-lift airfoils is approximately 0.0035. Hence if w is 14 lb. per square foot, V is approximately 64 miles per hour.

SECTION II

AVAILABLE MATERIALS

CHAPTER II

LOADS AND LOAD FACTORS

7. Loads Imposed on an Aircraft Structure.—It is quite obvious that, to design the structure of an airplane, it is necessary to know the loads which will be imposed upon the structure in order that we may determine the stresses in the members and hence determine the sizes of the members to carry the loads. A general classification of these loads is as follows:

1. *Air loads* on the main part of the structure, such as the wing and the fuselage, and upon the control surfaces, such as ailerons, elevators, and rudders.
2. *Static loads* other than air loads, such as loads in handling, taxiing, landing, etc.
3. *Dynamic loads*—any loads which involve the acceleration of the parts or of the complete airplane. These accelerations, of course, are encountered in maneuvers such as looping, spinning, diving, landing, etc.

8. Strength Criterion of Design.—What are the design conditions pertaining to air loads, static loads, and dynamic loads which would produce a satisfactory type of structure in the airplane? This is not an easy question to decide. As a matter of fact the question has never been definitely decided. It is probably the most discussed of all problems of design. The criteria set down by the Department of Commerce, Aeronautics Branch, the Army Air Corps, and the Navy Bureau of Aeronautics, were evolved after years of practical experience, industry, and careful consideration on the part of a great many aeronautical engineers; they probably represent a fairly accurate picture of the loads which are actually imposed upon the airplane in flight. These conditions number approximately 15 to 20. That is, to satisfy the government agencies, it is necessary to analyze an airplane structure for strength on the basis of approximately 15 to 20 conditions. The design of each member in the structure is ordinarily based upon the maximum load, positive or negative, which may be incurred by that member under any one of the conditions of operation. It is quite obvious, also, that,

if an airplane is designed for the exact condition of loading which may be incurred in flight, there may be a possibility of a slight weakness in some material of construction or a weakness due to corrosion or some other cause, so that the strength of the airplane may drop below the standard for which it is designed. Thus these conditions should also be provided for in the criterion of design.

9. Applied Loads.—Figures 5 to 13 show "free-body diagrams" of airplanes in various maneuvers and in various landing positions. These figures represent in general the usual fundamental design requirements specified by governmental agencies, whether in this country or abroad. It should be noted that, in general, insufficient data are available to determine the actual theoretical forces which may be represented in such a free-body diagram; hence, it is necessary to make reasonable assumptions, based on experience, in order that solutions of the applied forces may be obtained. Thus there will probably always be a difference between the theoretical and practical requirements.

An exact representation of the external forces acting on an airplane, assuming an application of d'Alembert's principle, requires that the conditions of equilibrium be satisfied. Considering the plane of symmetry of the airplane, we have

$$\Sigma F_H = 0 \text{ (horizontal forces)} \tag{4}$$
$$\Sigma F_V = 0 \text{ (vertical forces)} \tag{5}$$
$$\Sigma M_G = 0 \text{ (moments about the gravity axis)} \tag{6}$$

Radical assumptions concerning the magnitude, line of action, direction, and sense of applied external loads should be avoided in so far as possible; and *the equations of aerodynamics as well as those of equilibrium should be satisfied.*

Government requirements in regard to applied loads are varied and are constantly being changed; hence, it is impractical to attempt the presentation of detail requirements here. There are however certain standard fundamental concepts which the student should make a part of his thinking; these we present in the following paragraphs.

10. High Angle of Attack.—The act of pulling out of a dive with the airplane at the lowest point of the curvilinear path is generally referred to as representing the high-angle-of-attack condition. This loading condition deserves special consideration. Represent the forces as follows (see Fig. 5):

T = thrust.
L = lift.
D_p = parasite drag (resultant of all drag components excepting D_w).
D_w = wing drag.
P = down load on the tail (the horizontal component is absorbed in D_p).
$^{2}\theta/dt^{2}$ = angular acceleration of the airplane.
a_t = tangential acceleration.

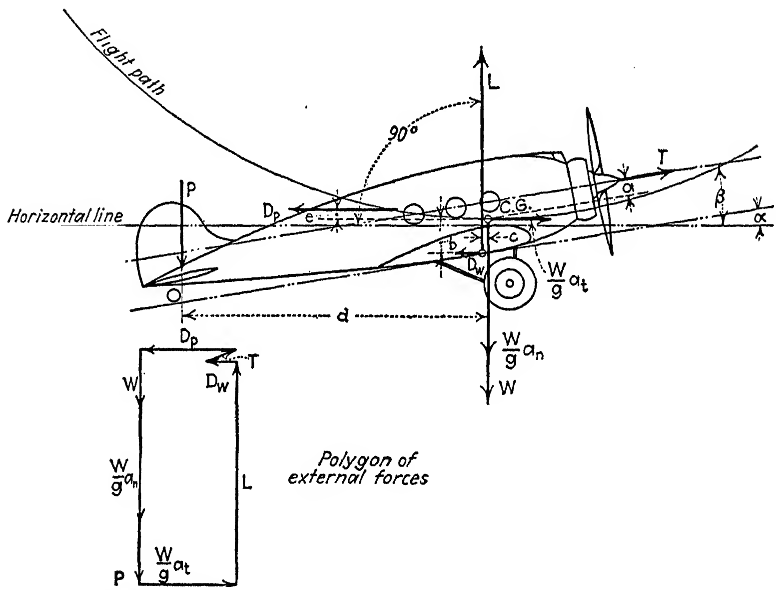


FIG. 5.—Diagram of forces acting on airplane in high angle of attack in curvilinear flight.

a_n = normal acceleration.
I = moment of inertia of the airplane about the c. g. axis.
N = load factor for high angle of attack.

$$(N - 1)W = \frac{W}{g}a_n.$$

With reference to Fig. 5, we may write the equations of motion as,

$$\Sigma F_V = 0 = T \sin \beta - [W + (N - 1)W] - P + L \qquad (7)$$

$$\Sigma F_H = 0 = T \cos \beta + \frac{W}{g}a_t - D_w - D_p \qquad (8)$$

$$\Sigma M_{cg} = 0 = Ta + Lc + D_w b - D_p e - Pd + I\frac{d^2\theta}{dt^2} \tag{9}$$

From these equations three quantities may be determined, such as T, L, and W.

At the present development of the method the usual practical solution of equations (7), (8), and (9) requires a number of assumptions. For example, we may assume, without serious error in most cases, that $\beta = \alpha$, that $I\frac{d^2\theta}{dt^2}$ may be neglected, that

$$\frac{L}{D_w} = K_{y(\text{max.})} = \text{constant} = k \tag{10}$$

and that

$$\frac{W}{g} a_t = D_p \tag{11}$$

We have from equation (8),

$$T \cos \alpha - D_w = 0 \tag{12}$$

Since, from equation (10), $D_w = L/k$,

thus

$$L = kT \cos \alpha \tag{13}$$

Substitution of this value in (9), gives us

$$Ta + kcT \cos \alpha + bT \cos \alpha - D_p e - Pd = 0$$

If we assume that $e = 0$, so that the parasite drag acts through the center of gravity, we may solve for P as,

$$P = \frac{kcT \cos \alpha + bT \cos \alpha + Ta}{d} \tag{14}$$

When substituted in (7), this gives

$$T \sin \alpha - NW - \frac{kcT \cos \alpha}{d} - \frac{bT \cos \alpha}{d} - \frac{Ta}{d} + kT \cos \alpha = 0$$

or

$$T\left(\sin \alpha - \frac{kc \cos \alpha}{d} - \frac{b \cos \alpha}{d} - \frac{a}{d} + k \cos \alpha\right) = NW$$

so that

$$T = \frac{NW}{\sin\alpha - \frac{a}{d} + \cos\alpha\left(k - \frac{b}{d} - \frac{kc}{d}\right)} \tag{15}$$

and

$$\frac{NWk\cos\alpha}{\sin\alpha - \frac{a}{b} + \cos\alpha\left(k - \frac{b}{d} - \frac{kc}{d}\right)} \tag{16}$$

and

$$D_w = \frac{NW\cos\alpha}{\left[\sin\alpha - \frac{a}{d} + \cos\alpha\left(k - \frac{b}{d} - \frac{kc}{d}\right)\right]} \tag{17}$$

It is to be noted, also, that the velocity may be evaluated from the relation (see Par. 6),

$$L = K_{y(max.)}AV^2 \text{ (engineering units)}$$

It is sometimes assumed that the propeller thrust is negative. This however does not alter the validity of the above analysis, *if the conditions of equilibrium are satisfied.*

11. Low Angle of Attack.—See Fig. 6. This is the condition of level unaccelerated flight at approximately maximum speed. The center of pressure on the wing approaches the rear spar. Equilibrium equations similar to those for the high-angle-of-attack condition may be written. The angle of wing chord to the path of flight may be zero, positive, or negative. The usual value of T used is the thrust required to overcome the parasite and wing drag at the angle of attack and speed corresponding to the center-of-pressure location. The tail load P may be calculated from the equations of equilibrium (for example, see Par. 105).

12. Inverted Flight.—See Fig. 7. The assumptions generally made for this condition are that the airplane is flying upside down at a high angle of attack, so that the center of pressure is forward. Since the chord component of the wing loading for the high- or low-angle-of-attack condition is greater than for the inverted-flight condition, the chord load for inverted flight is usually neglected in the analysis, that is, is assumed to be zero.

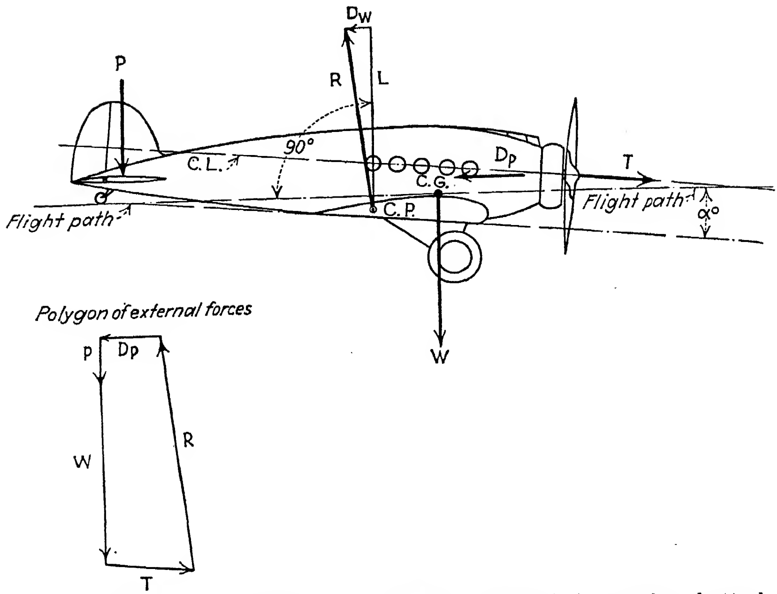


FIG. 6.—Diagram of external forces acting on airplane in low angles of attack, horizontal flight.

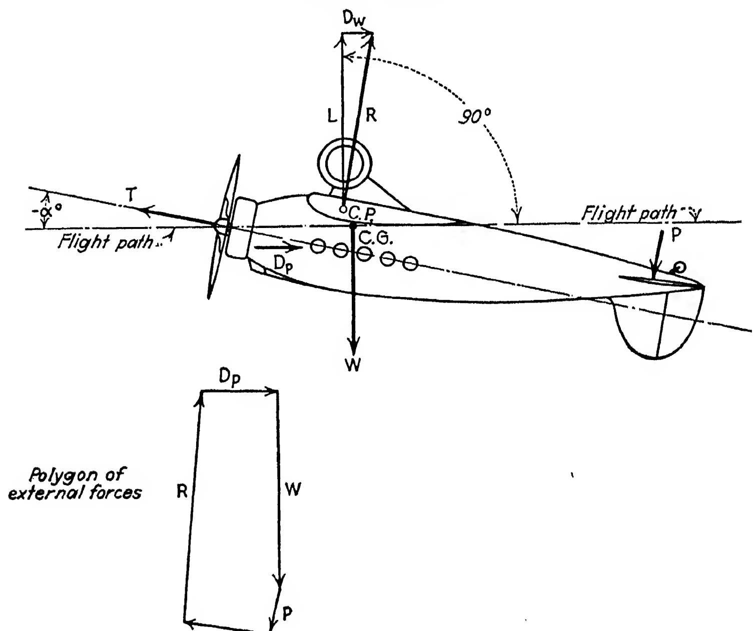


FIG. 7.—Diagram of forces acting on airplane in inverted flight.

13. Dive at Limited Velocity.—See Fig. 8. This is an assumed flight condition in which the propeller furnishes neither thrust nor drag. The velocity is limited to a specified value. The equations of equilibrium and of aerodynamics should be satisfied in the analysis.

14. Vertical Dive.—See Fig. 9. If we assume the thrust and the parasite drag to be zero in this case, P equals L. If these two

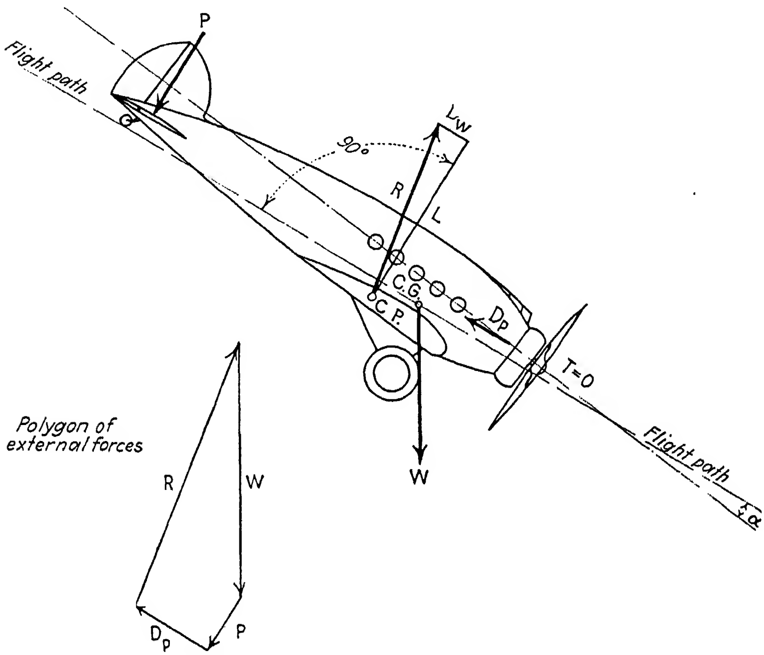


FIG. 8.—Diagram of forces on airplane in dive at limited velocity. (No thrust.)

quantities reach a high value, the moment about the wing axis becomes quite large. In this case, the weight of the airplane becomes the thrust, so that the drag of the airplane is equal to W. The equations of aerodynamics and equilibrium should be satisfied.

15. Three-point Landing.—See Fig. 10. It is generally assumed, in this case, that the airplane is resting on the tail skid or wheel and the two landing wheels. Applying the conditions of equilibrium, we note that

$$\Sigma F_V = 0, \qquad R_1 + R_2 = W$$

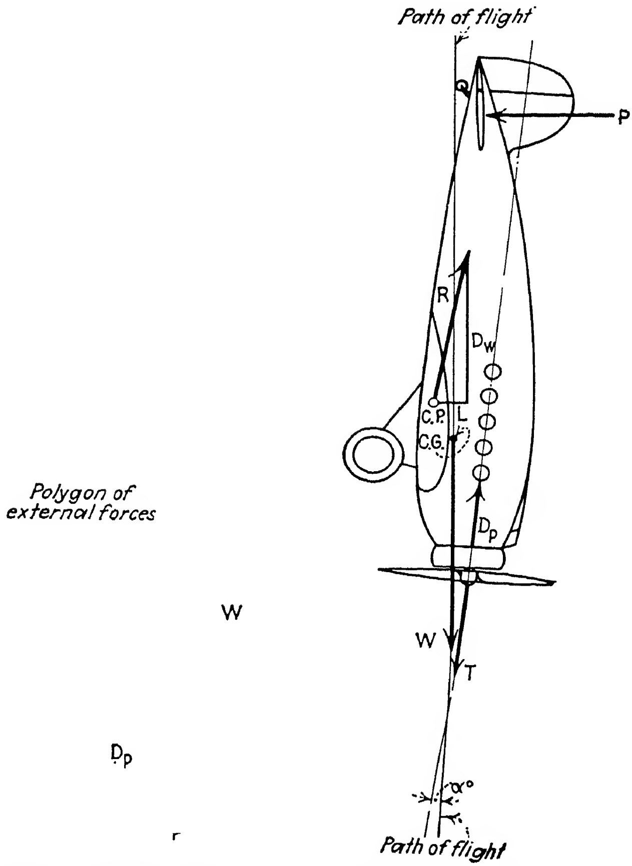


Fig. 9.—Diagram of forces on airplane in vertical power dive.

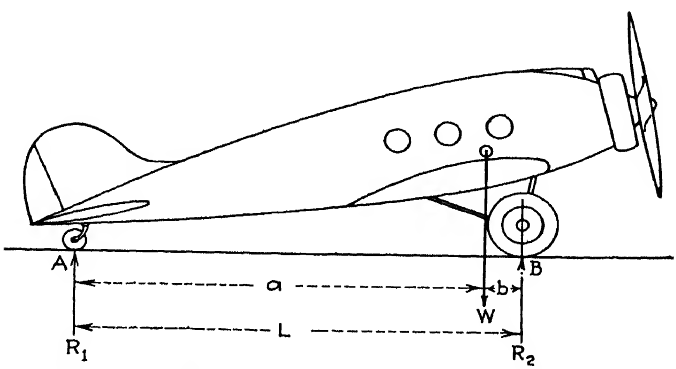


Fig. 10.—Free-body diagram of three-point landing.

and

$$\Sigma M_A = 0, \qquad Wa = R_2L$$

from which R_2 and R_1 may be obtained.

16. Level Landing.—See Fig. 11. This condition is sometimes called the "tail-high" landing. To satisfy the conditions of equilibrium, the resultant must pass through wheel axle O. The moment about this point is then zero. The force D may be the drag due to the brakes.

17. Nosing Over.—See Fig. 12. The statics of this condition is shown in the figure. We have

$$\Sigma F_V = 0, \qquad R_1 + R_2 = W$$

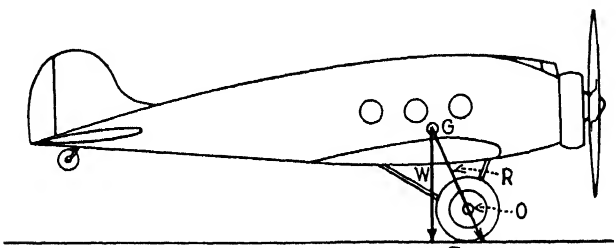


FIG. 11 —Free-body diagram of level-landing condition.

and

$$\Sigma M_A = 0, \qquad Wa = R_2L$$

from which R_2 and R_1 may be obtained. It is apparent that this maneuver involves certain problems in dynamics, but satisfactory relationships have not been determined.

18. Landing with Side Loads.—See Fig. 13. A "cross-wind" landing induces stresses not accounted for in any of the above-mentioned situations. The condition is a problem in dynamics. As the airplane touches the ground, the motion sideways is retarded. Assume the retardation to be an acceleration of a. A force $\frac{W}{g}a$ then acts at the center of gravity resisting the retarding forces, P and Q, on the wheels. A couple

$$(P + Q)h = \frac{W}{g}ah$$

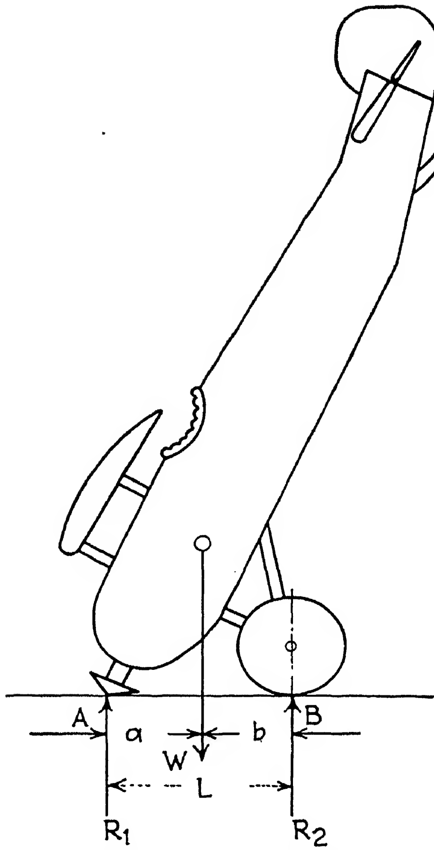
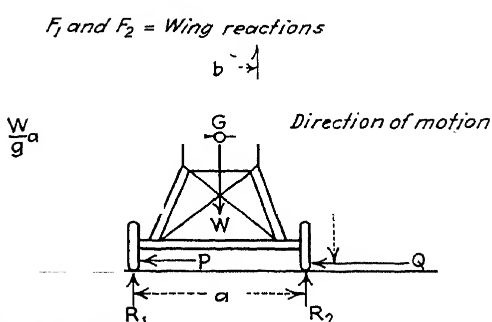


FIG. 12.—Free-body diagram of nosing-over condition.



$\Sigma F_V = 0, \Sigma F_H = 0, \Sigma M_G = 0$

FIG. 13.—Landing with side loads.

is thereby induced, tending to overturn the airplane. This couple is resisted by the couple $I_x\frac{d^2\theta}{dt^2}$, in which I_x is the moment of inertia of the airplane about an axis parallel to its x-axis (line of propeller thrust) and θ is the angle of rotation. The largest component of this couple is probably due to the inertia of the wings. It is $F_1b = F_2b$. If other components are neglected, the equations of equilibrium are

$$\Sigma F_H = 0, \qquad P + Q = \frac{W}{g}a$$

$$\Sigma F_V = 0, \qquad R_1 + R_2 = W$$

$$\Sigma M_G = 0, \qquad (P + Q)h = F_1\frac{b}{2} + F_2\frac{b}{2}$$

19. Other Types of Loads.—Other types of loads which may be required in a design are:

1. Unsymmetrical wing loading as in a "barrel roll."
2. Effect of air gusts.
3. Engine torque.
4. Stabilizer and elevator loads.
5. Fin and rudder loads.
6. Drop in a "pancake" landing.
7. Gyroscopic effect of propeller and engine.
8. Vibration of engine, propeller, or other parts.
9. Wing fluttering, tail fluttering, etc.

20. Load Factors.—In stress analyzing the average engineering structure, the maximum load which may occur on the structure is first determined. This maximum load is then multiplied by a certain *factor* which produces a *design load.* This factor is designated as the *factor of safety.* In aircraft work we have no such factor. If we should use a factor of safety, as defined, it is quite obvious that the airplane would be so heavy that it would not fly properly. Instead of a factor of safety in aeronautical engineering, we use what we call a *factor of loading.*[1] In using this factor of loading, the loads in the members are determined for normal maneuvers, that is, level flight or normal landing conditions. These loads are then multiplied by the *factor of*

[1] The industry generally considers the *factor of safety* to be the ratio between the maximum design load and the maximum probable load the airplane will encounter in flight for any given condition. Both the Navy and the Department of Commerce specify a working-load factor together with a factor of safety to be used in design.

loading to determine the *design load.* Load factors as used today are based mostly upon experience and very little upon theory. The problem of load factors is probably the most outstanding aeronautical engineering problem yet to be solved. The most fundamental question arising in the design of a new airplane is: How much stronger than the actual strength required for normal operating conditions shall we make the structural parts of this airplane?

21. Procedure in Finding Applied Loads.—An outline of the usual procedure in finding the loads on an airplane structure for design purposes is presented below. *This outline is not to be taken as a specific guide for a commercial stress analysis, but rather as a guide for the study of the general problem.* This outline of general procedure, however, follows very closely the usual plan for commercial analyses.

Outline of Procedure in Finding Applied Loads

A. Determine all load factors.
  1. Wings.
    a. High angle of attack.
    b. Low angle of attack.
    c. Inverted flight.
    d. Nose dive.
    e. Other conditions as required.
  2. Fuselage.
    a. High angle of attack.
    b. Low angle of attack.
    c. Inverted flight.
    d. Maximum stabilizer and elevator load.
    e. Maximum fin and rudder load.
    f. Level landing.
    g. Three-point landing.
    h. Landing with side load.
    i. Nosing over.
    j. Other conditions as required.
  3. Chassis.
    a. Level landing.
    b. Three-point landing.
    c. Landing with side load.
    d. Drop of specified height.
    e. Other conditions as required.
  4. Control surfaces and stabilizing surfaces.
    a. Specified maximum design loads.
  5. Controls.
    a. Loads produced by control surface loads.
    b. Other loads.

B. Determine design loads on wing ribs, wing spars, and wing-drag trussing.
   1. Find the distribution of the air load along the chord of the wing for the design conditions required.
   2. Find the division of the load between wings if more than one wing.
   3. Determine the effect of wing-tip losses, fuselage and nacelle interferences, effects of cutouts, etc., on the effective area (that area available for its full quota of lifting ability) of the wings.
   4. Find the effective area of each wing, or, if the chord is uniform in length, the effective span of each wing.
   5. Compute a gross loading curve for each wing—gross load per inch *versus* span in inches. In commercial analyses this is generally taken as the lift load, the total load being equal to the gross weight of the airplane.
   6. Compute a wing-weight curve—wing weight per inch *versus* span in inches.
   7. Compute a net wing-loading curve by subtracting the weight of the wing per inch from the gross loading per inch.
   8. Determine components of resultant load on wings in planes convenient for analyses of the structure, for example, the components perpendicular and parallel to the chord of the wing, called, respectively, the *beam* and the *chord* components.
   9. Apply load factors to these basic loads to obtain the design loads.

C. Determine the design loads on the fuselage.
   1. Distribute the weights such as those of the engine, passengers, gas, oil, structures, etc., as concentrated loads to points of application convenient for stress calculation.
   2. Compute the forces applied to the fuselage, such as those caused by the wings, the chassis, the engine torque, and the air loads for the several design conditions.
   3. The static loads on a fuselage should satisfy the conditions of static equilibrium, and the dynamic loads the conditions for dynamic equilibrium.

D. Determine the design loads on the chassis and tail wheel.

E. Determine the design loads on engine nacelles.

F. Consider control surfaces, control systems, and fittings.

22. Distribution of Air Loads.—The distribution and magnitude of air loads on wings, fins, and control surfaces for design maneuvers are, of course, the basis for design for flight conditions. An enormous amount of experimental work has been done to find a rational basis for determining the air loads. The student is advised to study reports on such air-load experiments, and to compare their findings with the government handbook requirements. The field of study is one which properly belongs to books on aerodynamics rather than to books on structures.

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CHAPTER III

GENERAL DESIGN PROCEDURE

23. Classification of Airplane-design Work.—Airplane-design work may be classified under three main heads, namely:

A. Design.
  1. Project design.
  2. Detailed design.
B. Stress analysis.
  1. Stress analysis of the primary structure.
  2. Stress analysis of secondary parts.
C. Drafting.
  1. Layout.
  2. Detailed.

Actual design work is generally carried out under the direction of a project engineer who is directly responsible to the chief engineer of the factory. Project-design work is usually handled by a graduate aeronautical engineer and includes not only structural design but also aerodynamic and economical design problems. The detailed design is usually performed by a draftsman qualified for this work. Detailed design consists of the design of small parts.

Stress analyses are usually performed by stress-analysis experts under the supervision of a competent engineer. Drafting, of course, is performed by competently trained draftsmen.

24. Specifications.—A proposed job of airplane design first comes to the attention of the engineer in the form of specifications. The proposed airplane may be of conventional design, or it may be new and a radical departure from conventional models. Thus a general outline of procedure may not be applicable in every case, but it will at least give some ideas of the steps which are to be followed in the design. The object of the airplane and its characteristics are set forth in the *specifications* for the job; for example, the order may be for a very-high-speed mailplane, a low-speed freight plane, a high-altitude plane, or a small sport plane. These specifications being assigned, we proceed as follows:

25. Structural Characteristics.—We determine the characteristics of the ship in regard to the structure; that is, we determine whether the airplane should be a biplane or monoplane, whether it is to be of wood and fabric construction or all metal, etc. These characteristics may also be specified in the order. When these characteristics are determined for the required specifications, we must then decide upon the power plant.

26. Power Plant.—Power plants are not designed for any specific airplane; consequently, we must buy our power plants to fit our job. Of all the power plants which are available we must determine the most desirable for the particular job, taking into consideration the characteristics of the power plant, the cost, etc.

27. Preliminary Sketches.—When the size and characteristics of the motor or motors are determined, then it is desirable to make a *preliminary sketch of the airplane*, incorporating all the principal parts. In this sketch the engineer should give careful consideration to the artistic appearance of the job. In general an airplane which is nicely streamlined and pleases the eye is also high in efficiency.

It is obvious that these preliminary drawings may be changed, and that we may not be able to build the airplane as pleasing to the eye as the preliminary drawings will indicate; we vary, however, as little as possible from the preliminary sketch of the job in order to incorporate the proper structural characteristics.

Obviously the structural and aerodynamic features must be satisfied, as they are of primary importance. The finished product should be a judicious compromise between the *artistic*, *aerodynamic*, and *structural*.

28. Preliminary Weight Estimate.—It is quite obvious that the strength of the structure will determine the weight of the structure. The weight of the structure, to a great extent, will determine the strength requirement. Each depends upon the other. There is no direct solution of the problem. At first we can determine only approximately what the weight of the structure will be; when we calculate the actual size of the members in the structure, so that we may determine the actual weight of the structure, we must revise our preliminary weight estimate. Now, for the purpose of making the preliminary weight estimate, we have the benefit of the experience of designers who have made a study of weight. For example, we have data available, assuming we are designing a conventional type of airplane, on

the proportional weight of the fuselage to the total weight, the proportional weight of the empennage, etc. Consequently we can form some judgment as to the proper distribution of the total weight of the airplane among the component parts. The total of these distributed weights equals the weight of the airplane as originally predetermined or estimated (see handbooks for weight estimates).

29. Airfoil-section Requirements.—With the preliminary weight estimate made, we can then determine the airfoil section required to carry this weight and execute the desired performance in regard to the speed, the lift, the rate of climb, etc. Having determined the airfoil section, we then carry out preliminary calculations for the dimensions of the wing, such as the aspect ratio, area of the wings, and other characteristics pertaining to the wing structure.

30. Preliminary Drawing.—We now make a preliminary side-view sectional drawing of the airplane for the purpose of obtaining the relative location of the airplane parts, cargo, and personnel; these will include the wings, engine, pilot, passengers, tail surfaces, landing gear, etc.

31. Preliminary Installation Drawing.—The next step is to complete the side-view drawing, in which we designate the location of instruments, gas tanks, fighting equipment (if it is a military plane), mail quarters (if a mailplane), etc.

32. Detailed Weight Estimate.—We are now ready for a more detailed weight estimate. In this weight estimate we include such things as the instruments and other small items in the airplane which, for simplicity in calculations, would not be included in the first preliminary weight estimate, except under general headings.

It is the usual procedure to increase the detailed weight estimate by about 5 per cent to account for small items neglected in the estimate.

33. Balance Analysis.—The center of gravity of the airplane must be located with a definite relationship to the center of lift of the wings. This requires a balance analysis. With reference to a V (vertical) and H (horizontal) axes coordinate system, with the origin fixed relative to the airplane, we may find by the application of the principle of moments $\bar{V}$ and $\bar{H}$, the coordinates of the center of gravity, thus

$$\bar{H} = \frac{\Sigma WH}{\Sigma W} \tag{18}$$

and

$$\bar{V} = \frac{\Sigma WV}{\Sigma W} \tag{19}$$

in which W is the weight of the part under consideration and H and V are the coordinates of the center of gravity of the part.

34. Revised Side-view Drawing and Balance Diagram.—We next revise the side-view drawing and balance diagram, more accurately locating the landing gear, wings, etc. It is apparent that, when the balance diagram, which also includes the wings and landing gear, is completed, it may be found more desirable to move the wings and landing gear than it is to move some part of the airplane structure or some part of the useful load. We may find it desirable to move the wings backward or forward slightly to insure that balance and stability are properly maintained. We mention the landing gear because it is obvious that the landing gear must be at least in front of the center of gravity of the airplane or the airplane would be inclined to land on its nose rather than on the two front wheels and the tail wheel; also, the center of gravity must be far enough behind the axles of the wheels to allow a reasonable braking force to be applied without nosing the airplane over. Yet we should not move the landing gear so far forward that the load on the tail wheel will be too great for practical purposes.

35. Design and Location of Ailerons and Tail Surfaces.—This problem is mostly a study in aerodynamics at this stage of the design. Data are available in texts and handbooks on the size, proportion, shape, and other features of airfoils affecting the stability and control of the airplane.

36. Final Three-view Drawing.—With the principal design details of the general layout determined, a final three-view drawing may be made. Certain complications may arise, however, which may necessitate later changes in these drawings.

37. Estimation of Performance.—With the weights, sizes, and shapes now practically fixed, it is desirable to make a careful check of the performance to see that the specifications have been met.

38. Determination of Characteristics of Wing Structure.—It is necessary to determine, first, the air loading on the wings in the required loading conditions, and then the loads in the assumed structural parts of the wing, before the exact design of the parts can be made. The general type of construction may probably be determined by specification. It remains for the designer to determine sizes and other details. This requires a stress analysis.

39. Determination of the Characteristics of the Fuselage Structure.—The general type of construction of the fuselage may be determined by specifications. It is probable, at least, that the type and general plan have been determined before this stage is reached. It is necessary, therefore, for the designer to determine sizes and other details. This requires a stress analysis.

40. Determination of the Characteristics of Tail-surface Structures and Ailerons.—In general, the types of these structures would have been determined previously; consequently, the sizes, special arrangements, and details only must be determined at this stage. This too requires a stress analysis.

41. Detail Design of All Structures.—The jobs may be listed as follows:

1. Wing ribs with fittings.
2. Wing spars with joints and fittings.
3. Trussing in the wings and between the wings.
4. Wing tips.
5. Fuselage bulkheads.
6. Fuselage, wing fittings, door and window reinforcements, pilot's compartment.
7. Ribs or longerons.
8. Stabilizer and fin.
9. Chassis and tail wheel.
10. Control surfaces: ailerons, elevators, rudders.
11. Control-surface controls with fittings.
12. Engine controls with fittings.
13. Electric wiring.
14. Passenger-compartment furnishing and equipment.

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CHAPTER IV

SELECTION AND CLASSIFICATION OF METALS FOR AIRCRAFT CONSTRUCTION. PROPERTIES

By N. F. Ward

42. General Requirements.—A well-balanced design of the airplane involves intelligent consideration of those properties, inherent in metals, which adapt them to structural requirements. Many metals are available for structural purposes, and their manifold characteristics, or properties, require careful consideration in order to produce the most suitable application in airplane structures. For an economical design, metals should possess as many of the following salient properties as may be obtained for any particular application:

1. Availability in quantity with reliability of source of supply.
2. Reasonableness in cost.
3. Physical properties to resist structural loads, such as tensional strength, stiffness, toughness, hardness, fatigue strength, and resistance, combined with light weight.
4. Mechanical properties for ease of fabrication, such as ductility, malleability, fluidity, low thermal expansion and contraction, and machinability.
5. Chemical properties for durability, such as weldability, corrosion resistance, and uniformity of composition, permitting design to close limits of strength, etc.

43. Special Metals.—The designer must be guided by careful study in procuring special metals or alloyed metals. This involves much attention to counsel of the experienced metal manufacturer or his agents, before restrictively designating any variety of metal for a special purpose. In seeking such advice, the designer must be guided largely by the experience of the airplane industry and of the skilled metal producer, not only in respect to the chemical composition establishing the variety of the metal, but also regarding the heat treatment of the material and a correct design which utilizes the metal effectively, as well as the metal's behavior in fabrication and flight.

44. Classification of Airplane Metals.—Metals for airplane construction may be classified in two general groups: (1) ferrous—those metals composed largely of iron, and (2) nonferrous—those metals constitutionally independent of iron.

45. Ferrous Metals.—The ferrous metals may be divided into the following groups:

1. Cast iron.
2. Wrought iron.
3. Steel.

Steel is the only one useful in aircraft structures.

46. Steel.—This metallic compound is produced by different processes by which it is classified, as *crucible, Bessemer, open hearth,* or *electric furnace.* The major distinction between cast iron and steel is the condition of the carbon. In steel the carbon appears entirely in the combined form (Fe_2C) in varying qualities, whereas in cast iron it appears as both combined carbon (Fe_3C) and graphite. Steel is also classified as *carbon* or *alloy steel.* Alloy steel has the specified carbon content and in addition other metals which noticeably affect the properties.

47. Crucible Steel.—This steel is commonly made in crucibles in small quantities of 90 to 100 lb. Low-carbon iron is soaked at melting heats in heating-furnace gases in the covered crucible. This process is used for high-grade tool steels and certain alloy steels.

48. Bessemer Steel.—This steel is made by blowing air through molten pig iron while in the Bessemer converter, as the container is called. The carbon is burned out along with some of the impurities. Addition of ferromanganese deoxidizes the molten contents of the converter in the ordinary or "acid" Bessemer process; the converter is lined with silicon-bearing earth which produces no effect on the phosphorus, thus producing an acid steel. In the "basic" process the lining is of magnesium compounds and limestone. The addition of the limestone during the refining of the pig iron keeps the slag basic. The phosphorus from the steel is absorbed by the slag and the lining, leaving a steel with low phosphorus content.

49. Open-hearth Steel.—Steel of this refinement is made by holding the pig iron in the molten bath of a regenerative furnace until carbon and impurities and other metals are reduced to the desired proportions. The process may be adjusted to produce

either basic or acid steel. The open-hearth process is more closely controlled than the Bessemer and produces very reliable steel. The open hearth produces five times more than the Bessemer in the United States.

50. Electric-furnace Steel.—This process resembles the crucible process except that the heat of the melting and refining is supplied by electricity instead of gas or oil fuels. The process can be readily controlled and produces high-grade alloy steels, either by the basic or by the acid method. This and the crucible procedure are the most expensive methods of producing steel.

51. Effects of Alloys in Metals.—When the addition of an element to a metal appreciably affects the properties of the metal, the element is said to be an *alloy*. Modern practice has been guided in the choice of metals by the desirable modification of their properties made possible by addition of these alloys. In order that we may more clearly understand the effects of alloys on properties of metals, mechanical working, heat treatment, and welding, we shall briefly consider a few of the principal alloys.

52. Carbon.—An increase of carbon in steels is accompanied by increase in hardness, yield, and ultimate strength. Increases in carbon content impart lower ductility, malleability, toughness, and impact resistance, and decrease weldability. High concentrations of free carbon at the surface of steel which is being welded decrease the fusion temperature required to effect a sound joint. This fact is made use of in conjunction with a filler rod of low-carbon content which, as the metal is deposited, increases in carbon content, producing a strong, ductile weld at a faster speed than is usually possible.

53. Chromium.—This element adds strength and increases the ductility and malleability of the iron carbide which is deficient in these properties. Additions of 1 to 2 per cent chromium produce a steel which responds to heat treatment and is of high strength, hardness, and fair ductility. Nickel-chromium alloys in steel may be heat treated to give ultimate tensile strength of 250,000 lb. per square inch with high ductility. Steels with 12.5 to 18 per cent chromium after proper heat treatment are resistant to corrosion and for this reason are called *stainless steels*. Hot working of these steels causes a loss of the stainless property. Steels which have approximate compositions of 1.75 per cent silicon, 8 per cent chromium, 22 per cent nickel,

1 per cent copper and less than 0.5 per cent carbon are stainless and heat resistant. Mechanical working increases their strength and hardness. Heat treatment (heating and quenching) does not harden alloys in this group. *The stainless steels, as a class, are nonmagnetic, giving immunity to radio interference.*

54. Manganese.—This is a valuable alloy in both steel and nonferrous metals. Manganese in steel reacts with sulfur and oxygen to produce clean tough metals. Manganese reduces brittleness at forging and at rolling heats. Up to 2 per cent of manganese in steel increases its strength and hardness and reduces its ductility when heat treated. Brittleness develops at between 1 and 7 per cent of this element. High manganese content, usually 12 per cent or more, produces a nonmagnetic, malleable, hard, strong steel which is tough and wear resisting and which has excellent fatigue strength because of its fine-grained structure.

55. Molybdenum.—This alloy has its best application in ferrous metals. Its presence in steel reduces grain size, and increases fatigue strength and homogeneity of the metal. Tensile strength, hardness, and resistance to impact and corrosion are also increased by its use. With chromium and carbon its effect is to intensify properties of metals with smaller additions than would be required of nickel or vanadium. To prevent extreme hardness in welds of chrome-molybdenum steel parts, moderate heats must be used to adjust weld and tube strengths to comparable values. Additions of molybdenum have been found to reduce scale formation on steels. Chrome-molybdenum steel can be heated, quenched, and tempered to develop tensile strengths varying from 90,000 lb. per square inch to 240,000 lb. per square inch with medium values of ductility. The ample supply of this element assures continued use of molybdenum as the principal alloy of aircraft steel.

56. Nickel.—This alloy is used in both steel and nonferrous metals. Nickel in steel develops greater strength, hardness, and toughness than does an addition of carbon. Without at least 0.25 per cent carbon the full effect of nickel cannot be obtained. Nickel strengthens the iron matrix, which is normally weak in plain carbon steels. It refines and toughens the grain of steel. Nickel reduces warpage and scaling of heated parts with the result that finishing costs are reduced. Combination of 8 per cent nickel or more with chromium (see Chromium) produces a

steel effectively resistant to corrosion at elevated temperatures; this is known as stainless steel.

Nickel as an alloy with aluminum strengthens and refines the grain structure. The fatigue strength and wear resistance are also increased by its use. These alloys are found in high-grade aircraft-engine pistons. Its use, otherwise, is limited at present.

57. Silicon.—This is a suitable alloy for steel, aluminum, and magnesium metals. Its addition serves to produce clean steel because it is a good oxygen scavenger (deoxidizer). Ductility of steel is improved with additions of silicon up to 2.5 per cent. Yield point and tensile strength are increased with additions up to 4.5 per cent. More than 6 per cent embrittles steel. A combination of silicon and manganese improves the impact resistance of steel. Silicon steels are quite malleable in the soft state and respond readily to heat treatment with only slight tendency to warp.

Silicon in aluminum and magnesium alloys assists the aging with consequent improvement in the strength of these alloys. In light piston alloys of these metals, it reduces chilling and subsequent embrittlement of hot metal. It acts as a scavenger of oxygen and producer of clean metal.

58. Tungsten.—The effect of this alloy is to produce the maximum hardness in steel. Toughness is increased by 0.3 to 2.25 per cent vanadium. Steels with tungsten, chromium, vanadium, and carbon, with occasional use of cobalt, molybdenum, or uranium, are called high-speed tool steels because of their ability to retain cutting hardness at red heats produced by high cutting speeds. These steels are expensive and require expert handling to develop maximum properties for machining operations.

59. Vanadium.—Vanadium is one of the few alloys in steel which increase the modulus of elasticity. It is a powerful deoxidizer in producing clean steel. The steel is materially toughened, hardened, and strengthened by its use as an alloy. Above 5 per cent, vanadium displaces the iron in the iron carbide increasing the toughness, fatigue resistance, and hardness of steels with slight loss in ductility. Vanadium steels can be forged, rolled, cold formed, and otherwise worked.

60. S. A. E. System of Numbering Steels.—The most common designations for steels is that developed by the Society of Automotive Engineers. For these steels the first digit has the following meaning:

1. Carbon steel.
2. Nickel steel.
3. Nickel-chromium steel.
4. Molybdenum steel.
5. Chromium steel.
6. Chrome-vanadium steel.
7. Tungsten steel.
8. Not yet assigned to any steel.
9. Silico-manganese steel.

For the alloy steels, the second digit signifies the average percentage of the alloying element. The remaining successive digits are the average points (0.01 per cent) of carbon content. For example:

Steel 1025 is a carbon steel with 25 points or 0.25 per cent carbon.

Steel 51236 is a chromium steel with an average of 12 per cent chromium and 0.36 per cent carbon.

61. Other Specification Standards.—Detailed specifications of metals are included in the "S. A. E. Handbook." Another basis of specifications is the U. S. Army and Navy standards. The U. S. Army Air Corps, the Navy, and the S. A. E. publish specifications for chrome-molybdenum steel tubing, and also for duralumin. The standard specifications are tentative until correlation of all essential requirements has been established by the Federal Specification Board in conjunction with the engineering societies, manufacturers, and other interested parties. These Federal Master Specifications as they are adapted will replace the U. S. Navy and U. S. Army specifications and become binding on all federal government departments.

Obviously the need of standard specification is important in order to determine the identity of the metal and its chemical and physical uniformity. The formulas as applied in design are based upon the assumption that the metal is uniform and homogeneous. The tests prescribed in the specifications are none too many for determining the quality of the metal the designer uses. Any lack of quality in the metal as received seriously impairs the safety of the airplane structure.

Tables I and II show the properties of steels used in the airplane industry with their designations according to either S. A. E. (Society of Automotive Engineers) standards or U. S. Army and Navy specifications.

TABLE I.—CARBON STEELS—ANALYSES, PROPERTIES, HEAT TREATMENTS, USES[1]

All These Steels Must Be Open-hearth, Crucible, or Electric-process Made

No	Analysis, per cent						Aircraft steels for	Heat treatment		Properties after heat treatment (minima)				
	C	Si	Mn	P, not over	S, not over	S. A. E. No.		Quenched in water at	Drawing temperature	Ultimate strength, lb./sq. in.	Yield point, lb./sq. in.	Elongation in 2 in., per cent	Reduction of area, per cent	Brinell hardness
1	0.05–0.15	....	0.30–0.60	0.045	0.050	1010	Deep drawing	Annealed, preferably in box		38,000–40,000	20,000–24,000	35	..	[2]
2	0.15–0.25	0.15–0.35	0.30–0.60	0.045	0.050	1020	Carburizing steel	1400–1425°F.	350–400°F.	of core 60,000				
3	0.20–0.30	....	0.50–0.80	0.045	0.050	1025	Fittings and misc. bolts and nuts	Cold rolled and annealed		55,000	36,000	22	..	[3]
4	0.30–0.40	0.15–0.35	0.50–0.80	0.045	0.050	1035	Structural steel	1525–1575°F.	1000°F. 1100°F. 1200°F.	103,000 100,000 95,000	70,000 65,000 60,000	25 30 32	63 66 68	200 195 180
5	0.40–0.50	0.15–0.35	0.50–0.80	0.045	0.050	1045	Cylinders, etc.	1475–1525°F.	1000°F. 1100°F. 1200°F.	115,000 110,000 100,000	80,000 75,000 70,000	20 23 25	55 58 60	230 215 200
6	0.90–1.05	0.15–0.35	0.25–0.50	0.040	0.050	1095	Springs	1400–1450°F.	650°F.					

[1] Courtesy of *Metals and Alloys*, compiled by Prof. Bradley Stoughton.
[2] Must bend cold on themselves without cracking.
[3] 180° cold bend around diameter = thickness of specimen.

Steel 1 Suitable for severe cold forming and deep cupping. U. S. Army Spec. 57-136-4.
Steel 2 Carburizing steel. Carburizes at 1650–1700°F. Cool in air. U. S. Army Spec. 98-10025 and 57-107-9.
Steel 3 For fittings and miscellaneous parts. Not suitable for deep forming or cupping. U. S. Army Spec. 57-136-3.
Steel 4 Structural steel for aircraft. U. S. Army Spec. 98-10025 and 57-107-20.
Steel 5 Cylinders, flanges, hubs, etc. U. S. Army Spec. 98-10025 and 57-107-15.
Steel 6 Spring tempered strip steel. U. S. Army Spec. 57-136-2.

TABLE II.—TERNARY AND QUATERNARY ALLOY, U. S. ARMY AIRCRAFT—STRUCTURAL STEELS ANALYSES, PROPERTIES, HEAT TREATMENT, AND USES*

U. S. Army Air Corps Spec.	Analysis, per cent				S. A. E. No.	Uses	Heat treatment			Physical properties (minima)				Brinell hardness
	C	Ni	Cr	Notes			Notes	Quenched in water (W) or oil (O), at °F.	Drawn at °F.	Ultimate strength, lb./sq. in.	Yield point, lb./sq. in.	Elongation, in 2 in., per cent	Reduction of area, per cent	
						Connecting rods			800	180,000	150,000	13	45	343
57-107-3	0.35–0.45	1.00–1.50	0.45–0.75	*a*	3140[aa]	Structural	Z	1475–1525 O	1000	135,000	115,000	17.5	57	277
						Crank shafts			1200	110,000	85,000	21	62	226
						Connecting rods			800	200,000	175,000	12	52	390
57-107-4	0.35–0.45	1.50–2.00	0.90–1.25	*a*	3240[aa]	Structural	Z	1475–1525 O	1000	155,000	130,000	15	56	320
						Crank shafts			1200	115,000	100,000	20	63	230
									400	260,000	150,000	8	38	540
						Connecting rods			600	240,000	175,000	10	42	510
57-107-6	0.45–0.55	1.50–2.00	0.90–1.25	*a*	3250[aa]	Structural	Z	1425–1475 O	800	230,000	190,000	12	46	460
						Crank shafts			1000	175,000	150,000	14	50	380
									1200	130,000	105,000	18	57	260
									600	225,000	130,000	10	40	440
57-107-7	0.30–0.40	2.75–3.25	0.60–0.95	*a*±	3425[aa]	Structural	Z	1450–1500 O	800	195,000	150,000	12	50	385
									1000	160,000	125,000	15	55	330
									1200	125,000	95,000	20	65	250
									400	220,000	110,000	10		
		V							600	200,000	125,000	10		
57-136-7	0.25–0.35	0.15–0.25	0.80–1.00	*b*	6130	Fittings and miscellaneous	Z	1600–1650 O	800	175,000	150,000	11		
									1000	135,000	110,000	12		
									1200	115,000	90,000	15		

57-107-2	0.30–0.40	0.15–0.25	0.80–1.00	*b*	6135	Connecting rods Structural Crank shafts	Z	1575–1625 O	400 600 800 1000 1200	250,000 230,000 200,000 150,000 120,000	130,000 175,000 175,000 115,000 95,000	12.5 12.5 12 15 20	43 45 48 50 55	460 420 400 350 240
57-107-5	0.45–0.55	0.15–0.25	0.80–1.10	*b*	6150	Structural	Z	1525–1575 O						
57-107-11	0.90–1.25	0.15–0.05	0.80–1.10	*c*	6195	Ball races and parts requiring maximum hardness and high bearing loads	Z	1425–1475 O	350					
57-180-2	0.25–0.35	Mo 0.15–0.25	0.80–1.10	*d*	X4130	Structural	Z Y	1600–1650 W	400	230,000 125,000	110,000 100,000	5 15		
10076	0.25–0.30	0.15–0.25	0.80–1.10	*d*±	X4130	Structural	Z	1600–1650 W	600 800 1000 1200	205,000 180,000 145,000 115,000	125,000 150,000 115,000 90,000	6 8 12 20		
57-107-19	0.25–0.30	0.15–0.25	0.50–0.80	*d*±	4130	Structural	Z	1600–1650 W						
	0.25–0.30	0.25–0.35	0.80–1.10			Structural								
	0.45–0.55	V 0.15–0.20	1.00–1.20	Mn 0.70–0.90		Chrome-vanadium spring steel	W	1550–1575 O	850–1050	190,000 and up	175,000 and up	11 and less	30 and less	363–444
	0.50–0.60	Si 1.80–2.20	0	0.60–0.90		Silico-manganese spring steel	W	1575–1600 O	850–1050	195,000–250,000	175,000–230,000	8–12	28–40	363–444

* Courtesy of *Metals and Alloys*, compiled by Prof. Bradley Stoughton.

a Mn = 0.30–0.60, P and S not over 0.040±, Si 0.15–0.35.

b Mn = 0.50–0.80, P and S not over 0.040±, Si 0.15–0.35.

c Mn = 0.20–0.45, P and S not over 0.030, Si 0.15–0.35.

d Mn = 0.40–0.60, P not over 0.040, S not over 0.045, Si 0.15–0.35.

aa OH, crucible or electric.

Z Normalized, annealed, then quenched.

Y U. S. Army Air Corps Spec. 57-136-8-allow mfg. to decide on this heat treatment.

W Annealed at 1600–1625°F.

62. Nonferrous Metals.—Under this classification are grouped the principal structural metals for aircraft, namely, aluminum and its alloys, magnesium and its alloys, beryllium, and copper-nickel alloys.

63. Aluminum.—Aluminum is extracted from an ore known as bauxite. Since the supply of bauxite is estimated as 8 per cent of the earth's surface, there is assurance of an ample supply.

Commercial aluminum is soft and has a tensile strength in the cast or annealed state one-fourth to one-fifth that of mild steel. The cast ingot is first broken down while hot and then treated by either hot or cold working. By cold working it is possible to double the tensile strength of pure aluminum with corresponding reduction in ductility.

64. Strong Aluminum Alloys.—Small impurities of iron and silicon of less than 1 per cent are sufficient to increase the tensile strength of pure aluminum 50 per cent and improve the hardness of aluminum alloys. Further additions of copper and magnesium produce an alloy susceptible to heat treatment which develops properties of strength comparable to those of mild carbon steel. For structural needs strong aluminum alloys are on a parity with many heat-treated steels because of lower specific gravity.

Aluminum-zinc alloys have a wide application in die castings. These alloys with large additions of zinc suffer reduction in tensile strength and embrittlement at high temperatures. Aluminum-silicon alloys make excellent castings which are hard, strong, and resistant to salt-water corrosion, but difficult to machine. Aluminum-manganese alloys produce a casting of low strength, but dense and tough. The ultimate tensile strength is not much better than 18,000 lb. per square inch. Aluminum-magnesium-silicon alloys react with suitable heat treatment, developing ultimate tensile strengths of the order of 50,000 lb. per square inch. Alloys of aluminum, magnesium, and silicon are readily cold worked but are not so strong as aluminum-copper alloys.

Alloy Nomenclature.—The nomenclature of the Aluminum Company of America for the wrought aluminum alloys is simple once it is understood. The symbol consists of three parts, for example: 17-S-T, written 17ST. The number 17 specifies the composition, the letter S signifies that the material is wrought (not cast), and the T denotes the temper. 17ST is known as *duralumin* and is the most extensively used alloy in the industry.

The wrought aluminum alloys may be divided into two classes: those that cannot and those that can be hardened by heat treat-

ing. The first class can be hardened only by working cold, while the latter can be hardened or annealed by the solution heat treatment. The various tempers are:

O—annealed, or dead soft.
H—hard rolled, hard drawn.
T—maximum heat-treated temper.
W—heat treated at room temperature.
RT—heat treated, aged and cold worked.
S—wrought (not cast).

Properties of strong aluminum alloys which are suitable for aircraft structures are tabulated in Table III according to their S. A. E. or U. S. Army designations.

65. Magnesium.—Magnesium as a metal for structural purposes is old in chemical history. But only recently has its use in structures been noticed. There is an abundance of magnesium in brine of sodium and magnesium chloride. Magnesium is extracted from this brine commercially by electrolysis. The extracted magnesium of 99.93 per cent purity is dipped from the top of electrolytic cells consisting of a rectangular cast-steel pot with 80 lb. brine capacity. The steel pot serves as the cathode and the graphite electrodes as the anodes. A current of 3,000 amperes is used. The process requires maintaining the magnesium chloride in the bath at the correct level.

66. Magnesium Alloys.—Magnesium as an alloy in aluminum has been referred to above. Other metals alloyed successfully and with noticeable effects on the properties of structural value are: aluminum, manganese, cadmium, and copper. The magnesium-aluminum-manganese alloys are used in castings for their superior mechanical properties, principally those of malleability and ductility. The magnesium and manganese alloys have low immunity to corrosion in salt water. Caution must be exercised in cleaning parts not to use caustic solutions such as oakite, wyandotte, etc., because all magnesium alloys deteriorate rapidly in these solutions. The magnesium-manganese alloy has excellent strength in wrought and heat-treated conditions, but is slightly inferior to the magnesium-aluminum-manganese alloys in this respect.

Forgings of magnesium combine light weight with high strength. Forging or hot pressing supplemented by heat treatment produces the best combination of strength and ductility

TABLE III.—ALUMINUM AND ITS ALLOYS FOR AIRCRAFT CONSTRUCTION

Reference	Chemical analysis, per cent									Name
	Al	Cu	Si	Mn	Mg	Zn	Ni	Fe	Total impurities	
U. S. A. 57-15-A	99.0[a]	....	Note *B*	.....	.....	.....	....	Note *B*	1.05+	U. S. Army Grade A—Note *Y*
U. S. A. 57-15-A	98.0[a]	....	Note *A*	.....	.....	.....	....	Note *A*	2.00+	U. S. Army Grade B
U. S. A. 57-15-1	99.5[a]	0.10[b]	Note *C*	.....	.....	.....	....	Note *C*	0.50+	U. S. Army Grade AA
Iron Age, Sept. 5, 1929, p. 615	97.0	....	....	1.25						
U. S. A. 98-10026	92[a]	3.5–4.5	....	0.4–1.00	0.2–0.75	.....	....	....	.....	Duralumin 17S[d] bar—Note *W*
U. S. A. 57-152	92[a]	3.5–4.5	....	0.4–1.00	0.2–0.75	.....	....	....	.....	Duralumin 17S[d] sheet[aa]
U. S. A. 57-187	29[a]	3.5–4.5	....	0.4–1.00	0.2–0.75	.....	....	....	.....	Duralumin 17S[d] tube
U. S. A. 57-153	92[a]	3.5–4.5	....	0.4–1.00	0.2–0.75	.....	....	....	.....	Duralumin 17S[d] bar
U. S. A. 98-10026	92.5[a]	4–5	1.20[b]	0.4–1.00	.....	0.25[b]	....	0.75[b]	.....	Duralumin 17S[d]
U. S. A. 57-72 Grade 6	87.0[a]	9.25–10.75	....	.....	0.35[b]	0.2[b]	....	1.00–1.50	.....	Duralumin casting
U. S. A. 57-72 Grade 3	92.5[a]	4–5	1.20[b]	0.35[b]	.....	0.25[b]	....	1.20[b]	.....	Duralumin casting
U. S. A. 57-72 Grade 1	90.0[a]	3.75–4.50	0.5[b]	1.25–1.75	.....	0.20	1.75–2.25	0.75[b]	.....	Casting—Note *Z*
"Strong Aluminum Alloys," Aluminum Company of America, 1928.	96.5[a]	....	1.0	.....	0.6	.....	....	....	.....	51S — 51-SO[e], 51-SW[e], 51-ST[e]
	95.0[a]	2.5	....	.....	0.3	.....	....	....	.....	A17S[d] Duralumin — A17SO[e], A17ST[e]
	94.0[a]	3.5	....	.....	0.3	.....	....	....	.....	B17S[d] Duralumin — B17SO[e], B17ST[e]
	93.0[a]	4.5	....	.....	.....	.....	....	....	.....	195 Casting
	92.0[a]	4.5	0.8	0.8	.....	.....	....	....	.....	25S[d] — 25SO[e], 25SW[e], 25ST[e]

TABLE III.—ALUMINUM AND ITS ALLOYS FOR AIRCRAFT CONSTRUCTION.—(*Continued*)

Reference	Chemical analysis, per cent									Name
	Al	Cu	Si	Mn	Mg	Zn	Ni	Fe	Total impurities	
	92.0[a]	4.0	1.25	0.5	0.5	.....	....	....	.....	C-17S (special 17S) C-17SO[e] C-17SW[e] C-17ST[e]
	92.0[a]	4.0	....	0.5	0.5	.....	....	....	.....	17E[d] Duralumin 17SO[e] 17ST[e]
Iron Age, Sept. 5, 1929, p. 615	92.0	4.0	....	0.5	0.5	.....	....	....	.....	Alclad—Note *X*
Bossert, *Trans. Am. Electrochem. Soc.*, September 1929.	92	8	....	.....	.....	.....	....	....	.....	No. 12 Casting
	95	....	5	.....	.....	.....	....	....	.....	No. 43 Casting
U. S. Army 98-10026	10.25–12.25	82[a]	....	.....	0.15[b]	0.25[b]	....	2.5–3.5	.....	Aluminum bronze
U. S. Army 98-10026	10.25–12.25	76[a]	....	.....	0.25[b]	0.25[b]	4.5–5.5	4.5–5.5	.....	Aluminum bronze

Courtesy of *Metals and Alloys*, compiled by Prof. Bradley Stoughton.

[a] Minimum. [b] Maximum. [c] Fe, Si, Mn, Cu, etc.—maximum. [d] The alloys called duralumin and alloy 17S are almost identical in limits of chemical analysis. [e] O indicates "soft temper." W indicates "as quenched temper." T indicates "heat-treated temper."

Note *A*—Si, max. 0.50. Fe + Si, max. 1.50.

Note *B*—Fe, max. 0.50. Fe + Si, max. 1.50.

Note *C*—Fe + Si, max. 0.40.

Note *D*—Aging effected by heating is called "precipitation heat treatment."

Note *E*—This is also called "solution heat treatment."

[aa] 57-152-2 Coated on both sides by pure aluminum. Excellent resistance to salt spray corrosion, but physical properties are slightly lower—called "Alclad." See Alclad below.

[bb] To anneal, heat these alloys to 650°F. and allow to cool. Special annealing required for 17ST—*i. e.*, heat to 800°F. and cool very slowly in furnace to 450°F.

[cc] If heated in a nitrate bath.

[dd] If quenched in cold water, the product has higher resistance to corrosion than if hot water is used.

25S for forgings.

25S Propeller blades. Will bend but not break.

51ST for crankcase.

Note *Z*—Used for pistons, air-cooled cylinder heads, bearing surfaces, and other high-temperature purposes. Specific gravity = 2.90.

Note *Y*—Aluminum tubes for gas, oil, and water lines, also for aluminum, gasoline, and oil tanks. May be enameled against hot water and gasoline. See U. S. Army Air Corps Spec. 3-135.

Note *X*—Alclad is used for wing coverings, sheet for dirigible gas bags, structural parts, and all parts requiring strength and resistance to corrosion.

Note *W*—Used for propellers, fuselage, and other highly stressed parts.

TABLE III.—ALUMINUM AND ITS ALLOYS FOR AIRCRAFT CONSTRUCTION.—(*Continued*)

Reference	Heat treatment						Physical properties				
	Annealing, °F.	Hardening—Note *E*			Aging—Note *D*						
		Temperature, °F	Soaking time, minutes	Quenching, water (W), air (A)	Temp., °F.	Time, hours	Tensile strength, minima	Yield point, lb./sq. in., minima	Elongation in 2 in., per cent	Hardness 500 kg., Brinell average	Shore minima
U. S. A. 57-15-A	....	....	......	....	....	....	12,000–16,000 22,000–30,000	....	30–45 4–1		
U. S. A. 57-15-A											
U. S. A. 57-15-1											
Iron Age, Sept. 5, 1929, p. 615.	....	....	......	....	....	....	15,000–18,000 27,000–35,000	....	15–30 4–1		
U. S. A. 98-10026	625–700	925–960	10–120	W at 60°F.	200–300	2	55,000	25,000	18	90	20
U. S. A. 57-152	625–700	925–960	10–120	W at 60°F.	200–300	2	55,000	30,000	12	....	20
U. S. A. 57-187	625–700	925–960	10–120	W at 60°F.	200–300	2	55,000	30,000	16		
U. S. A. 57-153	625–700	940–970	120	W at 60°F.	210–350	10–20	55,000	25,000	18	90	
U. S. A. 98-10026											
U. S. A. 57-72 Grade 6	....	940–970	240	A or W	385–415	1	30,000	....	......	100	
U. S. A. 57-72 Grade 3	....	940–970	16–24 hr.	W at 60–212°F.	285–315	2	30,000	....	3	70	
U. S. A. 57-72 Grade 1	....	940–970	240	A or W	385–415	1	32,000	....	......	90	
"Strong Aluminum Alloys," Aluminum Company of America, 1928.	650[bb]	970	25 ± [cc]	Cold W.[dd]	310–320	18	14,000–19,000	4,000–6,000	22–32	25–32	
	650[bb]	970	25 ± [cc]	Cold W.[dd]	310–320	18	30,000–40,000	15,000–20,000	20–30	55–70	
	650[bb]	970	25 ± [cc]	Cold W.[dd]	310–320	18	45,000–50,000	30,000–40,000	10–18	90–100	
	650[bb]	940–960	25 ± [cc]	Cold W.[dd]	....	....	20,000–25,000	....	20–28	30–40	
	650[bb]	940–960	25 ± [cc]	Cold W.[dd]	....	....	35,000–45,000	15,000–20,000	20–28	55–75	
	650[bb]	940–960	25 ± [cc]	Cold W.[dd]	....	....	20,000–25,000	....	20–28	30–40	
	650[bb]	940–960	25 ± [cc]	Cold W.[dd]	....	....	42,000–50,000	20,000–25,000	20–28	65–85	
	650[bb]	Al. Co. of Am.		HT No. 4	....	....	28,000–38,000	13,500	6–12	65	
	650[bb]	Al. Co. of Am.		HT No. 16	....	....	30,000–40,000	21,000	3–8	75	
	650[bb]	Al. Co. of Am.		HT No. 10	....	....	36,000–50,000	27,000	0–5	100	
	650[bb]	970	25 ± [cc]	Cold W.[dd]	285–295	8–15	23,000–35,000	7,000–12,000	12–20	45–55	
	650[bb]	970	25 ± [cc]	Cold W.[dd]	285–295	8–15	45,000–53,000	15,000–30,000	15–22	68–85	
	650[bb]	970	25 ± [cc]	Cold W.[dd]	285–295	8–15	55,000–63,000	30,000–40,000	16–25	90–105	
	650[bb]	920–940	25 ± [cc]	Cold W.[dd]	310–320	18	25,000–35,000	7,000–10,000	12–20	42–55	

TABLE III.—ALUMINUM AND ITS ALLOYS FOR AIRCRAFT CONSTRUCTION.—(*Continued*)

Reference	Heat treatment						Physical properties				
	Annealing, °F.	Hardening—Note *E*			Aging—Note *D*		Tensile strength, minima	Yield point, lb./sq. in., minima	Elongation in 2 in., per cent	Hardness 500 kg., Brinell average	Shore minima
		Temperature, °F.	Soaking time, minutes	Quenching, water (W), air (A)	Temp., °F.	Time, hours					
	650[bb]	920–940	25 ± [cc]	Cold W.[dd]	310–320	18	55,000–63,000	30,000–40,000	18–25	90–105	
	650[bb]	920–940	25 ± [cc]	Cold W.[dd]	310–320	18	63,000–70,000	50,000–55,000	8–14	95–125	
	650[bb]	920–940	25 ± [cc]	Cold W.[dd]	....	....	25,000–35,000	7,000–10,000	14–22	45–55	
	650[bb]	920–940	25 ± [cc]	Cold W.[dd]	....	....	55,000–63,000	30,000–40,000	18–25	90–105	
Iron Age, Sept. 5, 1929, p. 615	....	....	......	....	....	....	50,000–58,000	27,000–36,000	18–25		
Bossert, *Trans. Am. Electrochem. Soc.*, September, 1929.											
U. S. Army 98-10026	....	1650–1675	30–60	Cold water or brine	800–1000	....	75,000	40,000	......	3,000 kg. 200	
U. S. Army 98-10026											

Courtesy of *Metals and Alloys*, compiled by Prof. Bradley Stoughton.

[a] Minimum. [b] Maximum. [c] Fe, Si, Mn, Cu, etc.—maximum. [d] The alloys called duralumin and alloy 17S are almost identical in limits of chemical analysis. [e] O indicates "soft temper." W indicates "as quenched temper." T indicates "heat-treated temper."

Note *A*—Si, max. 0.50. Fe + Si, max. 1.50.

Note *B*—Fe, max. 0.50. Fe + Si, max. 1.50.

Note *C*—Fe + Si, max. 0.40.

Note *D*—Aging effected by heating is called "precipitation heat treatment."

Note *E*—This is also called "solution heat treatment."

[aa] 57-152-2 Coated on both sides by pure aluminum. Excellent resistance to salt spray corrosion, but physical properties are slightly lower—called "Alclad." See Alclad below.

[bb] To anneal, heat these alloys to 650°F. and allow to cool. Special annealing required for 17ST—*i. e.*, heat to 800°F. and cool very slowly in furnace to 450°F.

[cc] If heated in a nitrate bath.

[dd] If quenched in cold water, the product has higher resistance to corrosion than if hot water is used.

25S for forgings.

25S Propeller blades. Will bend but not break.

51ST for crankcase.

Note *Z*—Used for pistons, air-cooled cylinder heads, bearing surfaces, and other high-temperature purposes. Specific gravity = 2.90.

Note *Y*—Aluminum tubes for gas, oil, and water lines, also for aluminum, gasoline, and oil tanks. May be enameled against hot water and gasoline. See U. S. Army Air Corps Spec. 3-135.

Note *X*—Alclad is used for wing coverings, sheet for dirigible gas bags, structural parts, and all parts requiring strength and resistance to corrosion.

Note *W*—Used for propellers, fuselage, and other highly stressed parts.

obtainable in magnesium alloys (see Tables IV, V, and VI for composition and properties).

67. Tempers of Magnesium Alloys.—Magnesium sheet plate and strip, either rolled or extruded, are obtainable in the soft or hard tempers, indicated as annealed and hard rolled. Another temper is designated as HT or RT, designating heat treated and rolled after heat treatment respectively. Strong magnesium alloys harden rapidly with any type of cold working, that is, drawing, spinning, stamping, and general forming. In these instances it is advantageous to employ annealed sheet and to reanneal during mechanical working when the ductility and the available elongation become practically exhausted. This annealing temperature is approximately 660°F. Deformation in the annealing range requires less expenditure of energy than is required for hardened tempers. In the majority of instances, for fittings of usual design, forming is accomplished best in the annealing range. In the processes of spinning, bending, and stamping, the alloys should be brought to the working temperature before proceeding with the operations. This may be done by placing sheets or parts in 600-*W* oil at a temperature between 480°F. and 575°F. The oil heats the material uniformly and serves as an essential lubricant in the forming operations. Chilling reduces the plasticity of the sheet; hence it should be prevented during the cold forming by heating the tools or dies with a blowtorch or its equivalent to approximately 215°F.

68. Other Alloys.—There are many metals and their alloys which are structurally advantageous. Monel metal, an alloy of nickel and copper, has superior corrosion-resisting properties. Alloys of beryllium and copper, and beryllium and aluminum. have possibilities. But success is dependent upon suitable alloying to reduce extreme brittleness and excessive cost. Alloys of beryllium have been developed experimentally which show ultimate tensile strengths upon aging after heat treatment of 196,000 lb. per square inch and ductility as low as 3 per cent in a 2-in.-gage length. These alloys are passive to corrosion and, have high fatigue resistance.

69. Monel Metal.—This is a binary alloy of nickel and copper. The usual composition for airplane construction as found in the specifications of the U. S. Navy and U. S. Army Corps is nickel 68 per cent, copper 23 per cent, iron 3.5 per cent, maximum, and 2.5 per cent, minimum. Its use has been limited to fittings

TABLE IV.—CAST MAGNESIUM ALLOYS—PHYSICAL PROPERTIES*
(Jan. 24, 1933)

Alloy number	Condition	Tensile strength, lb./sq. in.		Yield stress,† lb./sq. in.		Elongation in 2 in., per cent		Average reduction in area, per cent	Endurance limit, lb. per sq. in. (500 million reversals)	Brinell hardness, 500-Kg. 10-mm. ball	Impact value (Charpy notched bar), ft. lb.	Modulus of elasticity, lb./sq. in.	Specific gravity	Old alloy number
		Average range	Minimum value	Average range	Minimum value	Average range	Minimum value							
AM240	As sand cast	22,000–24,000	22,000	10,000–12,000	9,000	1–3	1.0	....	.....	50–55	0.60	6,250,000	1.81	
	Heat treated only	30,000–38,000	29,000	12,000–13,000	10,000	6–9	6.0	....	.....	50–56	2.0			
	Heat treated and aged—1	30,000–33,000	29,000	15,000–17,000	15,000	3–5	3.0	....	.....	53–61	1.0			
	Heat treated and aged—2	30,000–34,000	30,000	17,000–19,000	17,000	1–3	1.0	....	.....	65–76	0.74			
AM241	As sand cast	23,000–25,000	23,000	11,000–12,000	10,000	3–5	3.0	2.0	6,000	44–48	0.8–1.0	6,250,000	1.79	AM7.4
	Heat treated............	29,000–36,000	29,000	10,000–12,000	9,000	6–10	6.0	6.0–10.0	7,000	46–56	1.8–2.4			
AM246	As sand cast	18,000–20,000	18,000	11,000–12,000	10,000	0–1	0.0	....	.....	65	0.37	6,250,000	1.82	AM12.4
	Heat treated	23,000–26,000	24,000	10,000–12,000	11,000	0.5–1.5	0.5	1.0	.....	64	....			
	Heat treated and aged—1	28,000–32,000	28,000	17,000–19,000	17,000	0–1	0.0	....	.....	76–80	....			
AM265	As cast	22,000–28,000	22,000	12,000–15,000	11,000	3–6	3.0	7.0	.....	53–57	1.07	6,250,000	1.84	AZG
AM720	As cast	18,000–20,000	18,000	8,000– 9,000	8,000	3–5	3.0	....	.....	42–45	0.40	6,250,000	1.85	
AM764	As cast	23,000–25,000	23,000	9,500–10,000	9,000	5–6	5.0	....	8,500	45–48	0.98	6,250,000	1.85	
	Heat treated	27,000–30,000	26,000	8,000– 9,500	8,000	6–9	6.0	....	.....	43–46	....		1.85	
	Heat treated and aged—1	29,000–36,000	29,000	18,000–24,000	18,000	3–5	3.0	....	9,000	60–70	0.75			

* Courtesy of Aluminum Company of America.
† Yield stress is taken as the stress on the stress-strain diagram which deviates 0.2% from the modulus line.

TABLE V.—WROUGHT MAGNESIUM ALLOYS—PHYSICAL PROPERTIES*
(Jan. 24, 1933)

Alloy number	Condition	Average tensile strength, lb./sq. in.	Average yield stress, lb./sq. in.†	Average elongation in 2 in., per cent	Average reduction in area, per cent	Endurance limit, lb./sq. in. (500 million reversals)	Brinell hardness, 500-kg. 10-mm. ball	Impact value (Charpy notched bar), ft. lbs.	Modulus of elasticity, lb./sq. in.	Specific gravity	Old alloy number
AM53S	Extruded (¾ in. dia.).	42,000–45,000	22,000–26,000	11–16	13–20	8,500	56–62	2.24			AM4.4
	Rolled—hard 0.125 in.	37,000	....	8	9	......	60	....			
	Rolled—annealed 0.125 in.	33,000	....	16	16	......	56	....	6,250,000	1.76	
	Rolled—hard, 0.051 in.	40,000–45,000	28,000–35,000	8–12	8	......	62	....			
	Rolled—annealed, 0.051 in.	34,000–36,000	18,000–21,000	6–9	26	......	56	....			
AM57S	Extruded rod and shapes	42,000–45,000	25,000–30,000	14	16–24	14,000	55	1.5–2.5			AZM
	Hot press forgings:								6,250,000	1.81	
	‡Heavy sections, ½ in.	38,000–43,000	20,000–28,000	5–8	.....	14,000	55–65	....			
	Light sections, ½ in.	43,000–45,000	27,000–31,000	10–14	.....	......	55–68	2.24			
AM58S	Hot press forgings	43,000–48,000	24,000–34,000	4–8	.....	......	70–76	1.35	6,250,000	1.83	AZ855
AM61S	Extruded rod (¾-in. dia.)	41,000–43,000	24,000–26,000	12–15	.....	13,000	54	....			
	Hammer forged	33,000–38,000	18,000–24,000	3–7	.....	9,000	45	2.0	6,250,000	1.85	
	Rolled—hard	35,000–38,000	20,000–26,000	5–10	.....	......	50	....			
	Rolled—annealed	30,000–33,000	18,000–20,000	8–12	.....	......	45	....			
XAM65S	Hammer forgings	37,000–39,000	20,000–22,000	9–17	.....	10,000	50–55	....	6,250,000	1.87	
XAM67S	Hot press forgings	38,000–40,000	18,000–21,000	16–20	.....	......	55–61				
	Heat treated and aged	35,000–30,000	22,000–25,000	1.5–4.0	.....	......	62–65	....	6,250,000	1.85	
AM3S	As rolled—hard 0.062 in.	35,000	27,000	6	.....	11,000	42	....	6,300,000		AM503
	Rolled and heat treated	28,000	14,000	14	.....	......	40	....	5,500,000	1.76	
	Rolled after heat treatment	35,000	25,000	6	.....	......	42				
	Hammer forged	33,000	16,000–20,000	7–10	.....	9,000	44	1.9–2.1	6,000,000		

* Courtesy of Aluminum Company of America.
† Yield stress is taken as the stress on the stress-strain diagram which deviates 0.2% from the modulus line.
‡ For sections having at least 80% deformation.

fuel lines, and tanks. It approaches copper in weight and costs about nine times that of mild carbon steel or approximately twice that of stainless steel. The minimum mechanical properties of standard products are compiled in Table VI and VII which have been furnished by the courtesy of the International Nickel Company.

Monel metal has an endurance limit in the annealed state of one-third its ultimate and in the cold-worked state, as in drawn tubing, of one-half the ultimate.

This alloy is immune to the atmospheric conditions encountered in flight and is heat resistant in so far as its strength properties are not appreciably affected up to 600°F.

The hardness of monel metal is varied from the annealed or dead soft condition to the hard-drawn state of high finish. Tubing is usually furnished in the "as-drawn" temper. Seamless tubing is furnished with annealed ends where severe expanding is intended. The sheets can be bent to the same limits as for stainless steel because of the comparable ductility.

Table VI.—Mechanical Property Ranges of Standard Products of Monel Metal

	Tensile strength, lb./sq. in.	Yield point, lb./sq. in.	Proportional elastic limit, lb./sq. in.	Elongation in 2 in., per cent	Reduction in area, per cent
Rod and bar:					
Cold drawn:					
Annealed......	70,000- 85,000	25,000- 35,000	20,000-30,000	35-50	65-75
As drawn......	85,000-125,000	60,000- 95,000		15-35	50-65
Hot rolled.......	80,000- 95,000	40,000- 65,000	25,000-40,000	30-45	50-65
Forged..........	80,000-105,000	60,000- 85,000	45,000-65,000	20-40	
Wire, cold drawn:					
Annealed........	70,000- 85,000				
No. 1 temper.....	95,000-110,000				
Regular.........	110,000-140,000				
Spring...........	140,000-175,000				
Plate, hot rolled....	60,000- 75,000	25,000- 35,000		25-35	
Sheet and strip:					
Full-finished sheet	65,000- 80,000	25,000- 35,000	20,000-30,000		
Cold rolled:					
Annealed......	65,000- 80 000	25,000- 35,000	20,000-30,000		
Full-hard sheet.	100,000-120,000	90,000-110,000			
Full-hard strip..	100,000-125,000	90,000-115,000			
Tubing, cold drawn:					
Annealed........	65,000- 80,000	25,000- 35,000	20,000-30,000		
As drawn........	90,000-105,000	60,000- 75,000		15-25	
Casting...........	65,000-100,000	30,000- 60,000		5-35	5-35

TABLE VII.—MINIMUM MECHANICAL PROPERTIES OF STANDARD PRODUCTS OF MONEL METAL

A. Cold-drawn Rod and Bar

Range of diameter or thickness, in.	Tensile strength, lb./sq. in., minimum	Yield point, lb./sq. in., minimum	Elongation in 2 in., per cent, minimum
Annealed:			
Rounds, squares, flats, hexagons, all	70,000	25,000	35
As drawn:			
Rounds:			
1/8 to 1/2	110,000	80,000	15
Over 1/2 to 1 1/4	100,000	75,000	20
Over 1 1/4 to 2	95,000	70,000	20
Over 2 to 3	85,000	60,000	25
Squares, hexagons, flats:			
Up to 1/4	90,000	75,000	20
3/8 to 1/2	85,000	65,000	25
Over 1/2	85,000	60,000	25

B. Hot-rolled Rod and Bar

Rounds:			
Up to 1/2	85,000	45,000	30
Over 1/2 to 1	80,000	40,000	30
Over 1 to 2	90,000	50,000	30
Over 2 to 3	85,000	45,000	30
Over 3	80,000	40,000	30
Squares, hexagons, flats, all sizes	80,000	40,000	35

C. Forged Rod and Bar

Rounds, rough-turned and specially straightened:			
Over 3 to 6	90,000	70,000	25
Over 6 to 9	85,000	65,000	25
Over 9 to 12	80,000	60,000	30
Squares, hexagons, flats:			
Up to 2	95,000	75,000	20
Over 2 to 3	90,000	65,000	25
Over 3 to 6	90,000	60,000	25

D. Cold-drawn Wire

Temper	B. & S. gage	Tensile strength, lb./sq. in.
Annealed	All	85,000 maximum
No. 1 temper	All	95,000 minimum
Regular	All	110,000 "
Spring temper	0– 2	140,000 "
Spring temper	3– 8	145,000 "
Spring temper	9–14	150,000 "
Spring temper	15–19	160,000 "

E. Plate: Sheet: Strip

Product	Tensile strength, lb./sq. in., minimum	Yield point, lb./sq. in., minimum	Elongation in 2 in.,* per cent, minimum
Hot rolled plate	60,000	25,000	25
Full finished sheet	65,000	25,000	35
Cold rolled sheet and strip:			
Annealed	65,000	25,000	35
Full hard	100,000	90,000	

F. Cold-drawn Tubing

Annealed	65,000	25,000	35
As drawn	90,000	60,000	15

* Values vary with thickness.

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CHAPTER V

MODIFICATION OF PROPERTIES OF METALS DURING FABRICATION

By N. F. Ward

70. Initial Conditions.—Metals as cast in the ingot or the mold are structurally useless even if chemically pure because of certain mechanical imperfections, such as:

1. Coarse crystalline structure which has poor fatigue resistance.
2. Nonuniform distribution of metallic grains or crystals due to uneven cooling. In other words, the metal lacks homogeneity.
3. Residual stresses due to uneven cooling from exterior to interior portions. Thus the load to which the metal may be subjected externally must be reduced by the value of these internal stresses.

71. The Production of Optimum Properties in Metals.—The coarse crystalline structure is the natural result in metals as they solidify from molten temperatures. Use of heat treatment alone does not reduce the crystalline structure. Only by mechanical treatment, such as hot rolling and subsequent forming coupled with heat treatment to relieve residual stresses, is the optimum uniformity obtained.

72. Mechanical Treatment of Metals.—The purposes of mechanical working of metals, in either hot or cold states, are, primarily, to reduce the heterogeneous metal to sound metal of uniform dependability and, secondarily, to produce useful structural shapes. The metal, in the form which the designer uses and specifies, is obtained by hot and cold rolling, extrusion, drawing, forging, or pressing, or by combinations of these mechanical operations. Heat treatment is a necessary supplementary operation to recrystallize or refine the crystalline microstructures and to reduce the residual stresses left from mechanical treatment. When mechanical working is done at temperatures below the recrystallization or annealing temperature, it is called *cold working;* when done above this range it is known as *hot working.*

Cold working results in a strain hardening of the metal. The metal becomes so hard that it is difficult to continue the forming process without softening by annealing. Structural sections such as rods (round, square, hexagonal, streamlined), tubes, channels, and other sections are made by hot and cold working. A typical example of mechanical working is the drawing of steel tubes.

The ingot from which the tube is to be drawn is first rolled into solid round bars approximately 6 in. in diameter and 42 in. long. These are then pierced axially at temperatures just below that of melting.

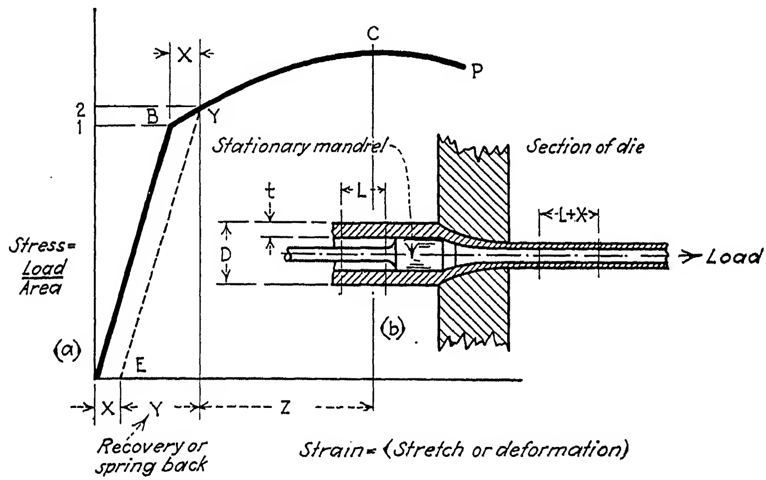

FIG. 14.—(*a*) Stress-strain diagram of low-carbon steel as annealed and cold drawn. (*b*) Cold drawing of tube.

After the pierced tube is formed and reduced to the smoothness desired by cross rolling and size rolling, the tube is cold drawn through white cast-iron or hardened-steel dies to form streamlined shapes or smaller sizes of tubing. It is not always possible to draw the tube into its final form in one pass through these dies because the metal is strain hardened and its ductility reduced.

In Fig. 14 is shown the tube in the drawing operation and its stress-strain curve. Examination shows that a unit length L has been elongated to a length $L + X$ while passing through the die over the mandrel. The tube has stretched permanently a distance X per inch of length at the expense of thickness t. If

the tube is in the annealed state and unloaded from the point B in a, it behaves elastically and returns to its former length L. However, by drawing the annealed or softened tube with a load exceeding that at B, the elongation increases plastically an amount X to an amount at Y. When the load is released, it recovers elastically an amount Y called "spring-back" to point E. The amount X is therefore a permanent deformation or elongation. This is substantially true except for a loss in mechanical hysteresis during recovery.

Comparing stresses B and Y, it is found that the cold drawing has elevated the yield-point strength with corresponding reduction in ductility before breakage, when reloaded. Reapplication of the load must exceed the value of Y before a permanent set develops. A load inducing a stress equivalent to C is accompanied by an additional permanent deformation of Z. Manifestly, cold drawing or cold rolling, which produces an equivalent distortion, results in a structurally useless tube, because the internal stress is at its ultimate value and a slight external load would result in rupture.

The drawing operation is carried to within 30 per cent reduction of area per pass which does not severely overstress steels. For further reduction in size or shape, annealing must supplement the process in order to relieve the internal stresses. With the ductility restored, further alterations in tube thickness and shape are possible. Stainless steel and chrome-molybdenum steel tubing are drawn for fabrication to a yield strength usually one-third greater than the yield strength for similar tubes in the annealed condition. With an elevated yield strength the tube reacts elastically until its cold-drawn yield-point strength is reached.

73. Defects in Mechanically Worked Metals.—Flaws in forgings, beside elongated blowholes, heat-treatment cracks, large slag inclusions, and laps are usually quite easily discernible. There are often to be found, by careful inspection, tiny cracks known as hair lines, which are about one-half to three-fourths of an inch long and from one to three thousandths of an inch deep. These hair lines or seams are undoubtedly nonmetallic inclusions of slag and manganese sulfide elongated by the rolling and forging operations. They originate in the cast ingot and are more numerous near the center in the usual method of casting. Some manufacturers do not use the center of the

ingot on account of the unsound conditions which generally exist there; the ingot is rolled into slabs and the center one is thrown out.

The only way definitely to avoid hair cracks is to produce a good sound ingot. Forging tends to diminish the seams rather than to increase them because the action results in a distortion and more complete interlocking of the crystals. However, it is impossible to interlock crystals which are held apart by slag and nonmetallic inclusions regardless of the amount of forging that is done.

Hair lines have no doubt always existed in steel, but they were either undiscovered or disregarded until the inspection became more rigorous on the material to be used for the construction of aircraft parts. It is quite rare to find a hair line which is not accompanied by others in the same region.

74. Effects of Heat Upon Cold-drawn Structural Shapes.—When the practical phases of construction are encountered, the cold-drawn shapes are subjected to heat such as in welding, bending, or straightening. The reversion of cold-drawn metals to their former annealed strength requires time. If heat is applied during short intervals, fabrication without causing recrystallization and relief of internal stress can be accomplished. Hard soldering is better for this joining of cold-drawn fuel lines or similar construction because it is done rapidly and at heat insufficient to anneal fully. Heat used in hard soldering, if localized, will relieve that section and produce warpage in cold-drawn material. This is to be expected because the heated area is softened, becomes more ductile, and absorbs the adjustment of the highly stressed, cold-drawn surrounding metal. Straightening of warped sections subjects the metal to cold working and may develop excessive internal stresses and cause rupture if carried too far. Uniform application and distribution of heat are requisite for success.

75. Influence of Welding Heat.—When stainless steel is welded, the metal undergoes fusion at high temperatures and the internal structure undergoes complex changes which are detrimental to corrosion-resisting properties. In chrome-molybdenum tubing or shapes the effects of heat are beneficial to certain sections and detrimental to other portions. Since the chrome-molybdenum steel hardens as it cools from fusion temperatures in the air, the strength of weld areas exceeds that

of cold-drawn sections. The fatigue resistance of the fused areas is reduced because of the large grain. The cold-drawn structure may be seen at the left of Fig. 15 gradually merging with the fused metal at the right. It should be noticed that there is a fine-grained section about half way between the cold-drawn and weld sections. On one side of this section the cold-drawn strength persists and on the other is an overheated portion which has cooled in air and consequently hardened to produce a static strength comparable with that of the cold-drawn portion. Figure 16 shows the usual type of failure in tension and its occur-

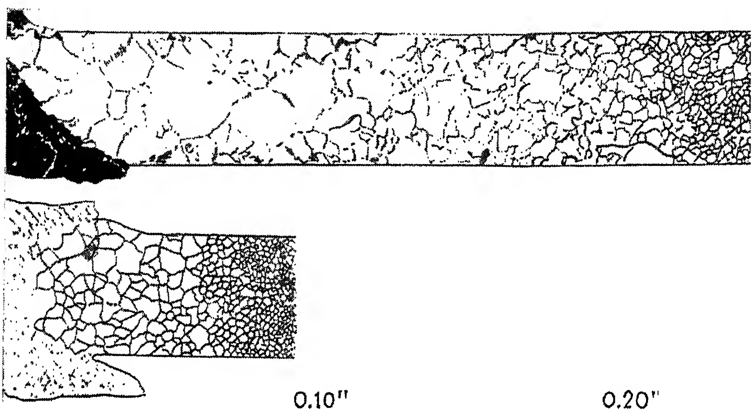

Fig. 15.—Thermal disturbance in oxyacetylene and electric welds. (*Courtesy of Westinghouse Manufacturing and Electric Company. From A. G. Bissell, Arc Welding in Airplane Industry.*)

rence at the point yielding initially, that is, the annealed section. At the left of the weld is another area, as far from the weld bead as the ruptured portion, which shows evidence of reduced section.

Welded tube sections, when properly heat treated by heating above recrystallizing temperature and quenched, develop uniform ultimate strengths as high as 110,000 lb. per square inch as compared with 94,000 lb. per square inch[1] for ultimate strength as welded. This heat treatment above the recrystallization temperature followed by quenching adjusts inequalities of crystalline matrix and ductility through the tube and joint. Excessive reinforcement beyond that specified for successful

[1] Johnson, J. B.: Welding in Aircraft, *Western Machinery World*, Vol. XXII, No. 10, pp. 451–453, October, 1931.

welds usually results in failure closer to the weld than previously mentioned.

76. Heat Treatment of Steel.—Heat treatment of metals may be described as consisting of a series of operations: (1) heating; (2) holding or soaking metal at elevated temperatures; and (3) cooling to room temperature, while the metal is in the solid state. The sole purpose is to alter the physical properties and to produce

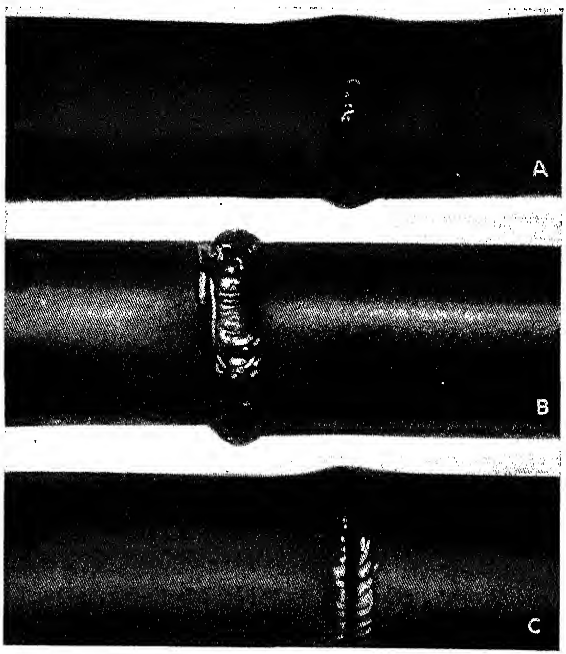

FIG. 16.—(*a*) Tension failure in oxyacetylene welded chrome-molybdenum steel tube. (*b*) Tension failure in arc welded chrome-molybdenum steel tube. (*c*) Fatigue failure in oxyacetylene flame welded chrome-molybdenum steel tube.

desirable physical changes in metals, that is, modification of strength, ductility, toughness and hardness, etc.

Constitutional changes in ferrous metals subjected to heat treatment occur at temperature levels depending upon the amount of carbon and alloys present. How dependent the response of steel to heat treatment is may be seen in Fig. 17. With additions of carbon, marked changes are noted after heating and quenching. These changes are due to the existence of allotropic forms of iron (or its alloys) and carbon. Investigation has shown that ferrous metals pass through changes of state in

the solid solution. In a sense water may be used as an analogy. In the ice state the water is solid, and on application of heat it liquefies at constant temperature (32°F.). With further heating, the water vaporizes at a constant temperature. Addition of another compound such as salt modifies these changes of state, causing them to occur at lower temperatures than for pure water. However, ferrous metals during heat treatment remain in the solid state and one must visualize the transformations occurring in that state. The iron at room temperature does not absorb

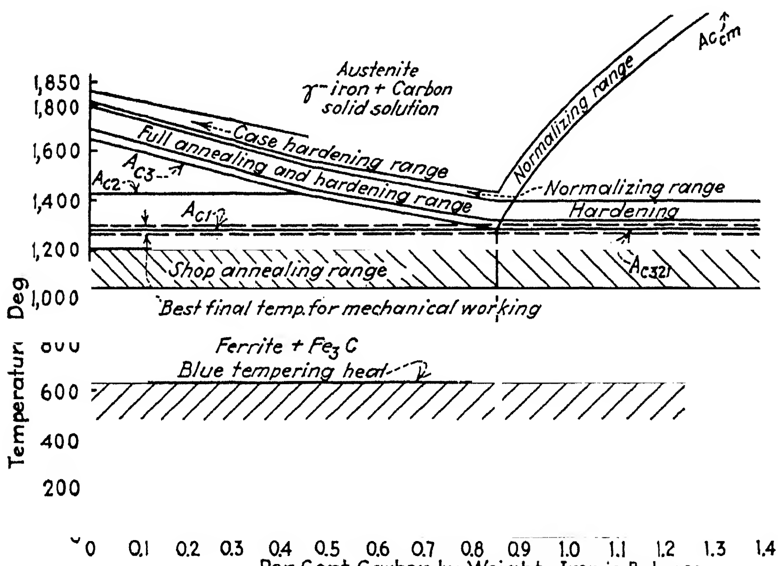

FIG. 17.—Thermal equilibrium diagram for iron-carbon alloys, showing normalizing, hardening, and tempering ranges.

carbon as water dissolves salt. At a higher temperature iron absorbs carbon and forms a solid solution. This combination is precipitated by rapid cooling. Properties exist which are not inherent in iron and carbon taken separately.

77. Mechanism of Heat Treatment of Steel.—When the ferrous alloys are heated, the allotropic changes are accompanied by a constant temperature condition. The temperature is a function of the amount of carbon present, as a consultation of Fig. 17 shows. The lags in temperature upon heating are designated as Ac (A = *arrêt,* meaning "lag" or "arrest period" in French, and c = *chauffage,* meaning "heating"). Corre-

spondingly, when the alloys are cooled, constant temperature levels are encountered which are termed Ar (r = *refroidissement*, meaning "cooling-back"). Thermal diagrams with the percentage of principal alloying element (carbon in this instance) are known as *thermal equilibrium diagrams*, and may be easily read for a known percentage of carbon when the terms Ac and Ar are understood.

The rate at which steel is heated or cooled determines the internal granular structure of the steel. The successive heat-treatment operations by which transformations are effected, are:

1. Heating the steel below or above the critical temperature.
2. Holding or soaking the steel at the specified temperatures long enough to heat the steel through and permit the change in grain size and allotropic form to go to completion.
3. Cooling the steel back to room temperature in the air, water, oil, or heated baths of oil, salts, or lead.

The airplane designer should have knowledge of such heat-treatment operations as annealing, normalizing, hardening, tempering, case hardening, and nitriding.

There are two kinds of annealing—stress-relief annealing and full annealing. *Stress-relief annealing* is accomplished by heating the work, usually welded parts of steel, until it shows a faint red color, which occurs at about 1100°F. (Fig. 17), and then cooling the heated parts in air. This operation removes lock-up stresses due to welding or cold working without changing the size of grain. The welding-torch flame is often used for this heat treatment.

Full annealing consists in heating the steel parts above the upper critical temperature range, holding the steel above this range until it is heated throughout and the grain has been refined, and then slowly cooling to room temperature. Full annealing of steel refines the grain of the metal, relieves locked-up stresses, and removes entrapped gases. Figure 17 shows the annealing range or the temperature to which steel must be heated and held before it should be cooled in air.

Normalizing is the process of heating steel above the critical temperature and then cooling below the critical temperature in still air. The normalizing range is made higher than the annealing range in order to obtain recrystallization of the steel without holding the steel so long at the temperature required for annealing. The normalizing range from which the steel is cooled is given in Fig. 17.

Another important operation is that of *tempering* which is carried on below Ac_1 (Fig. 17). The operation consists in re-treating hardened steels below the lower critical temperature and then cooling at a rate depending upon the alloy. The purpose of tempering is to reduce residual stress or hardening strains due to rate of quenching. It is attended by increase in toughness. Tempering is a necessary treatment of hardened parts to preserve maximum strength. For alloy steels, where toughness and strength are to be combined and yet supply sufficient ductility, tempering is desirable.

Hardening may be accomplished by heating and quenching iron-base alloys from above the upper critical temperature Ac_3 or Ac_{321}, for carbon steels above 0.85 per cent carbon content such as tool steels. For surface hardness, two methods of interest for use on airplane parts such as gears, axles, ball bearings, etc., are case hardening and nitriding.

By *case hardening* is meant the carburizing and subsequent hardening by suitable heating, quenching, and tempering of iron-base alloys. Carburizing consists in adding carbon to iron-base alloys by heating the metal below its melting point in contact with a carbonaceous material. This operation is the basis of case hardening. In this operation an outer case of high iron carbide content is produced without much effect upon the carbon content of the inner core. The process is usually applied to steels of less than 0.45 per cent carbon content.

Nitriding is performed by soaking special-alloy steels in anhydrous ammonia at temperatures below the lower critical temperature Ac_{321}. The hardness is attributed to absorption of nitrogen as iron nitrides by the steel. The outer surface is hard to less depth than the case-hardened steels. But the surface withstands higher temperatures and resists wear and corrosion better than does case-hardened steel.

78. Results of Overheating Steels.—Rapid heating usually warps steel, and therefore heat must be applied gradually. If the temperature is in zone 200°F. above Ac_3 or Ac_{321}, the crystalline structure enlarges and results in poor fatigue resistance. When the heat is excessive, the intercrystalline areas are burned or oxidized and consequently embrittled.

In alloy steels containing molybdenum and tungsten these metallic elements vaporize and are lost with excessive heat. The effects of these elements disappear as a result.

The microstructural changes with overheating of steel may be seen in Fig. 18. This photomicrograph of a tube shows an overheated surface of the tube at the left. In this area the grain has been noticeably coarsened by heating too far above the Ac_3 temperature. The grain boundaries have been burned and the enlarged grain has been deprived of its carbon by excessive heat. The structure to the right in this photomicrograph is of excellent refinement and distribution. In a surface as badly burned as this

FIG. 18.—Effects of overheating low-carbon steel contrasted with sound steel. Magnification 60 ×.

specimen, the strength has been decreased. Burned steel cannot be retrieved except by remelting. Precautions such as an accurate automatic temperature control during heating are of primary importance in preserving a sound, homogeneous steel.

79. Welding, Forming, and Deep Drawing of Stainless Steel. Welds of stainless steel are ductile without brittleness and are fairly homogeneous. Electric-arc, gas-flame, and resistance welding are equally successful. Although the physical properties are satisfactory, the recrystallization of the weld in localized areas reduces the corrosion resistance.

Annealed stainless steel responds readily in forming operations. Severe drawing is possible without intermediate annealing.

80. Allowable Safe-bend Radii.—A representative group of the stainless steels under trade names is listed in Table VIII.

TABLE VIII.—ALLOWABLE SAFE-BEND RADII FOR NICKEL-CHROMIUM STEELS[1]

Steel	Condition of steel tested	T Gage thickness, inches	R Minimum radius of bend, inches	Grain direction L = with grain C = cross grain	Ratio $\frac{I}{R}$	Safe bend radius recommended
Enduro-S stainless modified	Heat treated and polished. Hardened at 1750°F. Tempered at 800°F. Air cooled.	0.015 0.015 0.025 0.025 0.040 0.040	0.02 0.02 0.03 0.045 0.06 0.07	L C L C L C	1.3 1.3 1.2 1.8 0.5 1.75	$2T$ $2.5T$ $2.5T$
Alleghany stainless	Heat treated. Hardened at 1500°F. Tempered at 932°F. Air cooled.	0.015 0.025 0.040 0.040	0.030 0.050 0.070 0.070	C C L C	2. 2. 1.75 1.75	 $2.5T$
	Annealed	0.015 0.025	Close bend Close bend	C C	0 0	
5 % nickel	Heat treated. Hardened at 1700°F. Tempered at 800°F.	0.020 0.020	0.035 0.050	L C	1.75 2.5	 $3T$
2.8 % nickel	Heat treated. Hardened at 1750°F. Tempered at 800°F.	0.020 0.020	0.045 0.055	L C	2.25 2.75	

[1] Courtesy of H. S. Philips, Consulting Metallurgist.

81. Punching and Shearing.—As a further adjustment to the very great toughness and elongation of stainless steel, the engaging parts for shearing and punching must fit more neatly than for ordinary steel in order that clean edges may be secured; otherwise the metal may drag between the punch and die. Shear blades must press closely together for the same reason.

82. Drilling.—Sharp drills are required and must cut continuously to prevent hardening of the alloy under friction. High-speed tool-steel quality is required for this machining operation. The work has to be backed up so that the drill cuts through without punching out burrs. If a lubricant is required, lard oil and sulfur are effective.

83. Spinning.—The high ductility of stainless steel adapts it for spinning operations. Being a strong metal which develops greater stiffness with cold work, more power is required than for aluminum or brass. When total deformation exceeds that which can be accomplished in one operation, the work should be annealed in the manner given above. Thin sections employed in spinning can be heated by the electric resistance heater or in a muffle furnace for only about five minutes and then quickly cooled in air or in water.

84. Mechanical Treatment of Strong Aluminum Alloys. These alloys are available in the following forms: sheets, rods, and bar stock in drawn or extruded condition; extruded structural shapes, angles, T-channel, etc; tubular shapes, concentric or streamlined. The aluminum sheet is usually designated in the soft tempers for cold-forming operations.

The strong aluminum alloys having copper, magnesium, and manganese additions may be cold formed readily in the annealed state for the more severe bending operations. "Spring-back" often leaves the cold-formed parts out of alignment. This undesirable distortion can be removed by cold forming in suitable dies of hardened steel, bronze, or hardwood immediately after quenching in the solution heat treatment and before aging begins.

85. Forming Aluminum-alloy Sheets.—All forming on sheets should be done as soon as possible after quenching, as hardening (aging) begins almost at once. Aging can be retarded, however, by keeping the part below 32°F. Sheet, like rivets, obtains its maximum hardness in about four days.

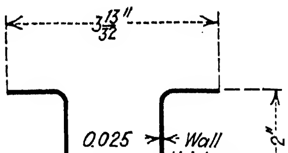

FIG. 19.—Cross section of longitudinal stringer.

Usual shapings are by hot forging or extrusion—as in the case of wires, rods, and structural shapes—cold pressing, stamping, and drawing. Cold drawing requires a lubricant such as paraffin oil to retain smoothness and to reduce power. Many of the longitudinal stringers (Fig. 19) and open shapes such as rib sections (Fig. 20) are drawn directly from strip stock of aluminum alloys in the dead soft temper without reannealing.

Punching with accurate clearance between die and punch gives satisfactory holes when the sheet is supported with hard-backing-up material so that tearing at the bottom of the hole

does not occur. Shears and snips distort the edges of the sheet, producing a lateral flow which is difficult to remove. If the distortion is permanent, a poor fit results. Use of a band saw with coarse blading or a nibbler produces the least distortion and insures the best alignment of adjacent seams or parts.

86. Prevention of Surfaces Abrasion or Notch Effects on Aluminum.—A smooth uniform surface can materially increase the life of these alloys. A soft lead pencil should be used for marking layout on sheets in preference to scribers or center

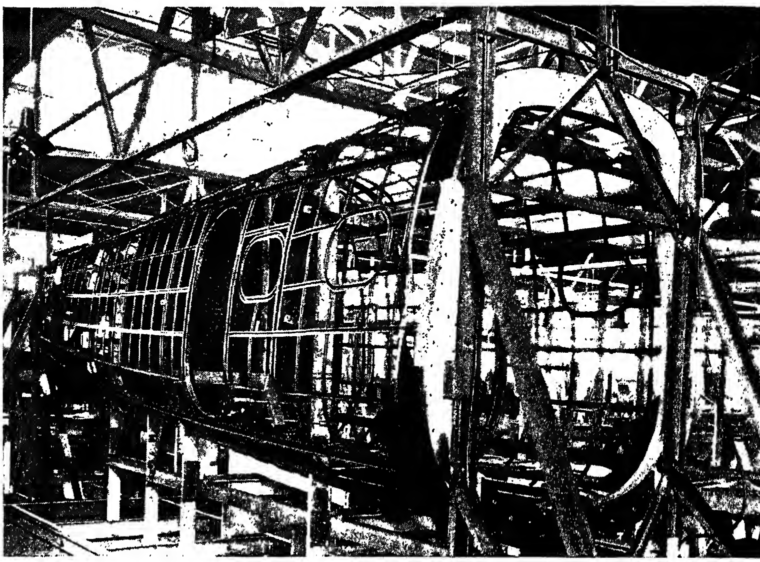

Fig. 20.—Fuselage construction of Boeing Transport (247). (*Courtesy of Boeing Airplane Company.*)

punching. When center punching is required for locating holes which will be punched through later, a light tap of the punch will prevent distortion of the sheet. Grit in forming dies produces stress inflation and rough surfaces which "anti-sieze" compounds will not prevent. Soft clean metal such as copper should be used on the face of regular vises in which aluminum alloy is filed. A rough surface on machined alloys results from lack of cutting compound and insufficient cutting speed. A mixture of 50 per cent lard oil and 50 per cent kerosene prevents adherence of aluminum to cutting-tool edge, a condition which usually causes a rough surface.

Bending has been mentioned above. We note that the amount differs with each alloy. To preserve maximum design strength, it is necessary to adhere to the radii of bending determined for alloys. Various aluminum alloys differ in ductility and as a result cannot be bent over the same bend blocks and still retain the same radius of bend. Consequently the forming radii for various tempers vary for the same alloy. Sometimes the actual values used differ from one shop to another. Table IX giving approximate bend radii, is included here as a guide for the designer. Cold forming of the hardest tempered alloys 17SRT and 24SRT requires the largest radii. Investigation indicates that forming of heat-treated alloys immediately after quenching (within an hour) does not markedly produce harmful mechanical properties such as a decrease in resistance to corrosion.

TABLE IX.—180° COLD BEND OF ALUMINUM-ALLOY SHEET[1]

Alloy and temper	Bend radii for sheet (Brown and Sharpe gage)			
	26	20	14	
2 SO..........	0	0	0	0
2 SH..........	$\frac{1}{2}t$	$\frac{1}{2}t$	t	$1\frac{1}{2}t$
3 SO..........	0	0	0	0
3 SH..........	$\frac{1}{2}t$	$\frac{1}{2}t$	t	$1\frac{1}{2}t$
17 SO..........	0	0	$\frac{1}{2}t$	t
17 ST..........	$\frac{1}{2}t$	$\frac{3}{4}t$	t	$1\frac{1}{2}t$
Quenched 17 ST.	$\frac{1}{2}t$	$\frac{3}{4}t$	t	$1\frac{1}{2}t$
Alclad 17 ST....	t	$2t$	$2t$	$2t$

t = thickness of sheet.
[1] Courtesy of Aluminum Company of America.

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CHAPTER VI

PROTECTION OF METALS AGAINST CORROSION

By N. F. Ward

87. Provision for Inspection, Cleaning, and Draining.—Wear, fatigue, and corrosion are the major enemies of the airplane structure. Only through periodic inspection may harmful inroads of these enemies be determined. The problem of the designer is to provide adequate inspecting hatches in readily accessible locations. Provision for removal of accumulations of dirt must also be considered, since its lodgement in crevices enables it to absorb moisture on which corrosion thrives. Moisture-proof joints are vital in corrosion prevention.

Provision must also be made for drainage of water from the structure.

88. Corrosion of Metals.—Corrosion may be defined as the intercrystalline action, resulting in loss of metal, that transpires when a metal is exposed to varying atmospheric conditions.

Iron is the principal element in steel used in airplane structures, and aluminum is the principal element in strong aluminum alloys. Like other elements, iron and aluminum have the property of going into solution only by displacing some other element already in solution. For example, a piece of iron, placed in copper sulfate solution goes into solution, but at the same time copper is plated out of the solution and appears as a coating on the iron. In the usual case of iron present in water, hydrogen is the element plated out. This hydrogen collects on the iron in the form of a thin, invisible film.

The presence of the hydrogen film tends to obstruct the progress of the reaction by insulating the metal from the solution since the hydrogen is insoluble in the metal. This interference may be so effective in pure water as to stop corrosion. This is the first stage of corrosion.

For corrosion to continue, the hydrogen film must either escape as gaseous hydrogen or combine with dissolved oxygen, which is usually present in water solutions, to form water or insoluble

hydroxides. The corrosion process is then free to continue so that more iron goes into solution, more hydrogen plates out, and the process continues at a rate depending upon the speed with which oxygen removes the hydrogen. This is the second stage in corrosion, and it accounts for the continuance of the process and gradual pitting of the surfaces. The iron, however, goes into solution as a rust which coats the iron and may form a protecting medium that interferes with the corrosion reactions by insulating the metal from the solution. The same reactions take place in acid solutions but at accelerated rates. The hydrogen gathers so rapidly that it is forced off in the form of gas bubbles as in a storage-battery solution being charged. In badly segregated iron and steel, and in other metals containing impurities which are foreign to their composition, corrosion is more rapid than for homogeneous, clean metals. A similar result is noticeable where metals with electropositive or electronegative characteristics are in contact.

This discussion does not apply to stainless steels and various alloys of nickel, chromium, and silicon. These steels do not accelerate corrosion under conditions which would make iron vulnerable.

Unprotected aluminum alloys on wing covering or fittings react rapidly in salt air because of the mild acid formed as a coating.

89. Corrosive Resisting Coating.—Prevention of premature failure from corrosion in metal parts constitutes an important phase in preservation of metal. Protective coatings are found in three classes: (1) paints, (2) greases, and (3) metallic films.

The primary purpose of applying protective coatings is to stop corrosion in its first stage. Wood is never allowed to contact metal members as it has residual moisture which makes the steel favorable to corrosion. Dissimilar metals such as magnesium alloys and duralumin should be kept from coming in contact with each other because there are feeble electric currents set up which cause the metals to go into solution.

All metal parts in an airplane are subject to corrosion and must be protected on both inside surfaces and external surfaces.

90. Paints.—Some paints such as asphaltum and tar protect the surface of metal merely by the formation of an impervious film. Other paints exert a chemical protective action. These are represented by linseed-oil-base paints mixed with pigments of red lead, oxides of iron, aluminum oxide, lead sulfate, graphite,

hydrocarbons, or other combinations. The paints must be inactive with adjacent metals. The pigments serve this purpose. The linseed oil in paints absorbs oxygen from the atmosphere and forms an elastic bond with the metal as the oil thickens. This reaction between linseed oil and oxygen is accelerated by boiling linseed oil (called *boiled oil*) before using and adding salts of manganese or lead which are called *driers*.

Paints serve to protect the surface from dampness, sea water, oxidizing gases, smoke, etc. Periodic inspection of painted surfaces is necessary to check surface cracks, peeling, or spalling, which leaves the surface vulnerable to corrosive action.

Paints are applied by spraying, by brushing, or by dipping.

91. Greases.—Greases and oils are temporary preventives of corrosion on finished machine parts, tools, etc., while they are being shipped or stored or otherwise not in service. The better grades of greases and oils possess the following characteristics:

1. Film coating is nonporous, uniform, and tenacious.
2. Film is stable chemically and is resistant to normal temperature fluctuations.
3. Film does not crack or dry out. Air-proof and moisture-proof paper is used as wrapping material for oil-coated pieces or parts to prevent drying out of the oil and to exclude the agents necessary for corrosion, principally water and oxygen.

Oils and greases may be applied by brushing, dipping, slushing, or spraying. During wet-grinding operations, rust-preventive oils have been mixed with the cutting fluids to reduce subsequent corrosion.

92. Metallic Films.—Metallic coatings may be applied to ferrous metals by dipping parts to be protected into hot baths of the coating metal, or by electroplating with a metallic coating from an appropriate solution. Hot dipping is less expensive and faster, but as yet is not so effective as electroplating. The more common processes of metal coating are galvanizing, Sherardizing, Parkerizing, Coslettizing, tinning, and chromium, nickel, cadmium, and copper plating. These methods pertain to the machine industries dealing with manufacture of machine parts and equipment, and will be outlined briefly below.

For the metallic coating to adhere firmly, the surfaces to be coated must be clean and free from all foreign material such as grease alkalis, soap, etc.; and the coating must be deposited free from strains by using moderate heats in order that strains will

not be severe enough to crack the surface and allow corrosion to develop. Subsequent heating and cooling may be necessary to leave the coating and the metal mutually devoid of strains. Tests on chromium-plated measuring gages have demonstrated that reheating to 570°F. and cooling after electroplating resulted in the maximum wear and abrasive resistance.

93. Galvanizing.—The process of coating metal with a thin layer of zinc is known as *galvanizing*. When the galvanized part is exposed to corrosion, the zinc is dissolved or goes into solution instead of the iron, since zinc is of higher potential electrochemically and hence is more active than the iron.

The part to be galvanized is dipped in a bath of molten zinc (hot galvanizing) at a temperature of from 800° to 925°F. Cold galvanizing is done by electroplating zinc from a solution of zinc sulfate and cyanide as the electrolyte. The part being plated is attached to the negative electrode (cathode) and the metallic zinc is dissolved from the positive electrode (anode).

94. Sherardizing.—The objects to be Sherardized are placed in a revolving drum or retort with zinc oxide dust. The retort is closed and heated to about 700°F., which is below the tempering temperature of the steel being Sherardized, for one-half hour to several hours. Sand is often added to the retort charge to prevent caking of the zinc dust and to brighten the coating. This coating is not suitable for stay wires or members subjected to bending.

95. Parkerizing and Coslettizing.—Parkerizing and Coslettizing differ only in time of heating and elements used. In Parkerizing, the objects treated are soaked in a bath of about 2 per cent phosphoric acid, manganese dioxide, etc., heated above the boiling point. When effervescence of the bath ceases, the work is removed and dipped in oil. The surface appears as gun metal. In Coslettizing, the work is treated in a bath of 1 per cent phosphoric acid and a small quantity of iron filings for one-half hour to three hours. These operations are only slightly effective in warding off corrosion of steel.

96. Hot Dipping of Cadmium and Tin.—Cadmium coatings are obtained by dipping parts to be coated, after cleaning, into melted cadmium at 700°F. or by applying the coating of cadmium with a steel-wire brush at 660°F.

Steel sheets are dipped into molten tin which is held at temperatures between 480° and 600°F. The steel sheets are care-

fully cleaned in two operations: the first is termed *black pickling* and is carried out on the sheets of steel in a dilute solution of sulfuric acid (4 per cent strength) which is kept hot with live steam circulating through the bath. After annealing, cold rolling, and a second annealing, the sheets are given another dilute sulfuric acid (2 per cent strength) immersion which is subsequently washed off. The sheet is then dipped by passing it through palm oil to prevent oxidation of tin on the sheet and is immersed in a flux of zinc chloride $ZnCl_2$; this is followed by soaking in molten tin, where the coating of tin adheres to the steel.

97. Electroplating.—The material to be plated (copper, nickel, brass, aluminum, steel) is held at the negative terminal (cathode) in the plating bath (electrolyte).

Chromium plating is not satisfactory on zinc, aluminum, and cadmium. An effective means of cleaning steel plate preparatory to chromium plating is to make steel the anode in a chromic acid bath. The more durable chromium plating is accomplished by plating with copper or nickel first, then following with the second plating of chromium.

98. Finish for Internal Surfaces.—After the steel fuselage is welded, the interiors of the tubes are oiled throughout, holes $\frac{1}{16}$ in. in diameter having been drilled near each joint for this purpose. Lukewarm lionoil, linseed oil, or varnish is forced through the $\frac{1}{16}$-in. holes with an average pressure of 10 lb. per square inch, which forces the oil into the inner recesses of the joints and surfaces of the tube. The penetration of the oil can be detected by feeling the warmed tubes. Any welds which are porous permit the warm oil to seep through and be noticed. Defective welds can then be repaired before the exterior of the tubes is coated. The oil is permitted to drain out after these operations and the holes are sealed by welding to exclude any moisture.

99. Corrosive Resistance of Stainless Steel.—Metallurgy has not yet produced a metal or alloy equally adaptable to all purposes in airplane construction. The development of chromium and iron alloys with or without nickel has made available for structural design a class of metals combining unusual resistance to corrosion, ease in fabrication, durability against wear, good fatigue resistance, and other desirable physical properties.

The distinctive property in the stainless steels of resisting corrosion is attributed to the formation of characteristic surface

film on clean metal when chromium is condensed in quantities of 11 per cent or more with the iron in low-carbon steel. Corrosive action usually requires the presence of oxygen and it is conceived that the film produced on high-chromium irons generates a continuous, stable invisible film which acts as a barrier against further corrosive action of external agents. The existence of this protective film is dependent upon the presence of oxygen.

100. Corrosion of Aluminum Alloys.—Corrosion of aluminum structures is nearly always, directly or indirectly, caused by moisture. Moisture not only will corrode duralumin but furnishes a medium for the electrolytic action of two adjacent dissimilar metals. The remedy for corrosion, then, would be to eliminate all moisture. This is impossible in aircraft structures, which are subject to varying climatic conditions, so other methods must be resorted to.

Causes for corrosion difficulties in strong aluminum-alloy structures are one or more of the following:

1. Misapplication of unprotected aluminum alloys to salt-water uses.
2. Inadequate provision for protecting aluminum alloys in service.
3. Improper maintenance of units in service.

101. Corrosion Prevention in Aluminum Alloys.—Experiments of the Aluminum Company of America and numerous investigators have demonstrated that dry aluminum does not corrode. The effect of thickness of metal has no bearing on the ability to resist corrosion except that deterioration resulting from attack produces a greater percentage of loss in the mechanical properties of the thin sections.

Design methods for combating corrosion of strong aluminum alloys may be listed as follows:

1. In design, water pockets should be avoided and provision be made for drainage, especially in pontoons.

2. One of two alternatives should be provided, either watertight assemblies or accessibility for inspection and maintenance.

3. Alloy with the highest passivity and with an adequate strength-weight ratio should be used. Installation of metal in the best corrosion-resistant state is advisable.

4. Surface preparation must be selected which adequately prepares the metal prior to riveting. Chromic acid-anodic treatment is the best method yet developed. This anodic oxidation treatment is essentially electrolytic, the parts being the anodes and the bath of relatively pure chromic acid and water the electrolyte. Degreasing of the surfaces to be anodized is required and final anodization is done at bath temperatures between 100° and 110°F.

After one hour of immersion at impressed voltages varying from 0 to 50 during the hour, the chromic acid must be washed from the parts with hot water.[1]

In order to produce a smooth uniform surface which acts as an excellent base for paints, anodized parts can be drawn through smooth dies in cold forming without destroying the surface coating of chromic oxide.

5. Aluminum paint has given excellent results on anodic aluminum alloys. Bituminous and asphaltum-base paints serve well for submerged parts in the absence of anodic coatings.

6. Parts disposed to collect and hold water should be painted before assembly, or the joints should be made watertight.

7. Where steel is adjacent to aluminum, aluminum foil should be interposed to prevent contact of surfaces. In general unlike metals require pure insulators such as foil to arrest any galvanic action.

8. Screws and rivets should be coated with fresh paint where practical.

9. Use of aluminum foil instead of dope-proof paint is promising as an effective corrosion deterrent.

10. Certain alloys (AM61S, Aluminum Company of America designation) have high corrosion resistance except in salt water where special surface protection is required. Protection is obtained by washing in a solution of acid sodium dichromate (6 lb. per gallon), one gallon 60 per cent nitric acid, 3 gallons of water, followed by final washing in hot water. After this has dried, an application of good paint such as aluminum paint or varnish produces immunity to corrosion. As with all paint coatings, frequent inspection must be made to detect surface cracks to forestall corrosion. Washing of magnesium in lye or caustic solutions such as oakite, wyandotte, etc., precipitates the magnesium and brittle powder, causing the pure alloy to waste away. Gasoline and glycerin soap have no appreciable effect upon these alloys and are good cleansing agents.

11. *Alclad* is a trade-mark for heat-treated strong aluminum alloy (17ST) which is protected from corrosion by smooth dense surface layers of relatively pure aluminum alloyed with the core. The passivity of pure aluminum coating is one explanation of the corrosion resistance of Alclad. Another reason is that the higher solution potential of pure aluminum causes electricity to flow from the coating core. Tests[2] show that when Alclad (12ST) sheets are joined by regular 17 ST rivets the surface attack occurs very noticeably at the points on the aluminum coating adjacent to the rivets. Tension tests show no reduction in strength after this action takes place.

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[1] "Aluminum in Aircraft," pp. 102–123.

[2] "Aluminum in Aircraft," Aluminum Company of America, p. 100.

SECTION III

BASIC STRUCTURAL ANALYSIS

CHAPTER VII

STATICALLY DETERMINATE STRUCTURES

102. Equations of Static Equilibrium.—It is possible by the method of statics to solve for a maximum of six unknown values of loads in a structure. Let us consider the fundamental principle involved in the solution of problems by statics. Fundamentally, the method of statics is an analytical one. Our graphical methods are applicable only to special cases of statics. Let us first consider the fundamental equations of static equilibrium. With reference to a rectangular coordinate-axis system of notation, the equations specify that the components of the forces parallel to the axes must be in equilibrium, and the components of couples about the axes must be in equilibrium. Thus we write:

$$\Sigma F_x = 0 \tag{20}$$
$$\Sigma F_y = 0 \tag{21}$$
$$\Sigma F_z = 0 \tag{22}$$
$$\Sigma M_x = 0 \tag{23}$$
$$\Sigma M_y = 0 \tag{24}$$
$$\Sigma M_z = 0 \tag{25}$$

In these equations x, y, and z refer to coordinate axes which are perpendicular to each other. Any other set of axes with the same characteristics would serve as well. The moment equations may be varied to suit the problem; that is, the axes x, y, and z, in so far as moments are concerned, may not necessarily be the same as the axes x, y, and z for the forces.

103. Application of Equations of Equilibrium.—To solve any problem by statics we proceed as follows:

1. Draw a "free-body" diagram of the complete structure as a unit, and then a free-body diagram of each one of the members of the structure as a unit. *A free-body diagram* is the diagram of a structure or a member of a structure in which the outside members that react on the structure or the member in question are removed and vectors representing the forces transmitted by those members are substituted in their places.

2. The next step is to write the equations of equilibrium for the free-body diagram of the complete structure and for each member of the structure. The components of forces parallel to the axes are used.

3. The next step is to incorporate the geometrical and trigonometrical relations and other relations in these equations of equilibrium.

4. We then solve the equations of equilibrium for the unknown quantities.

104. Free-body Diagram.—If one is proficient in drawing free-body diagrams of a structure or the members of a structure, one will usually find very little difficulty in the remainder of the solution of a problem. Some of the essential features of the free-body diagram are:

1. Extreme care should be used in drawing the vectors for the *outside* reactions. The vectors must indicate the forces *applied by the outside members* and not the forces applied by the member itself.

2. In case the weights of the members of the structure must be considered, the manner in which this may be done so as to cause the least confusion is always to assume the weight of a member as an outside force. This means that we must consider all our members as weightless, then apply the weight of the member as a vector of an outside force either at the center of gravity or in components, so that the resultant of the components of weight passes through the center of gravity.

3. One mistake which is commonly made in drawing a free-body diagram is as follows: After the convention of *direction* and *sense* has been assumed on the outside of the structure or on some particular member, the definition of the convention is violated on some other member. For example, if member A reacts on the member B with a force of 10 lb. to the left, then obviously, we assume the member B reacts on the member A with a force of 10 lb. to the right. The selection of the sense of the vector is entirely arbitrary for the first member considered. However, the selection of the sense thereafter, in connection with any of the other members, must conform to the selections already made.

4. If a load is applied at the intersection of two or more members, it is sometimes confusing to determine just how we are to apply this load. Quite often students apply the load to each one of the members. Obviously this is erroneous because the load really is impressed upon the structure only once. If we apply it to all the members at the particular joint, it is equivalent to applying just that many more loads. The desirable procedure to follow in this case is to assume that the load is applied to one member infinitely near the pin at the joint.

5. In *determining the sign of the forces or of the moments* in methods of statics, probably the least difficulty in guarding against errors is incurred by following the convention of signs adopted for the free-body diagram rather than by following the mathematical convention of signs for the direction cosines. This means that one should constantly bear in mind the physical problem as pictured by the free-body diagram, and should constantly observe that the conventions of signs of the free-body diagram are

adhered to in the equations. This will save considerable difficulty in the final equations. It implies that all direction cosines are considered positive. This system is adhered to in this text.

105. Example of Equations of Equilibrium. *Example* 1.—In analyzing a fuselage structure for stresses in the low-angle-of-attack conditions, it is necessary to find the down load P on the horizontal stabilizer. The wings are generally analyzed in terms of the actual weight W of the airplane, rather than in terms of the lift L, which is the sum of P and W. Assume a monoplane (Fig. 21), for which we have determined the spar reactions on the fuselage in terms of the weight of the airplane. Assume the center of drag of all parts other than that of the wings to be on

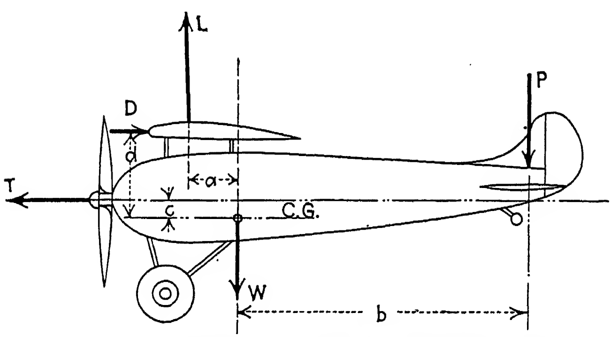


FIG. 21.—Equilibrium of airplane in low angles of attack. Dimensions a, c, and d, may be negative or positive. Dimension a, in general, is toward the rear from $C.G.$

the line of the propeller thrust. Derive equations for determining a correction factor for the spar reactions with a wing load of L, and for determining P in terms of the wing drag D, the weight W, and the necessary dimensions.

Neglecting drag of all parts other than that of the wings, we may write the following equations of equilibrium:

$$\Sigma F_x = 0, \qquad T = D \tag{26}$$
$$\Sigma F_y = 0, \qquad L = P + W = KW \tag{27}$$
$$\Sigma M_0 = 0, \qquad aL + bP + dD - cT = 0 \tag{28}$$

We have, combining (27) and (28),

$$a(P + W) + bP + dD - cT = 0 \tag{29}$$

or

$$(a + b)P = cT - aW - dD \tag{30}$$

From (26),

$$T = D$$

Thus (30) becomes

$$(a + b)P = cD - aW - dD = (c - d)D - aW$$

Hence,

$$P = \frac{(c - d)D - aW}{a + b} \tag{31}$$

Substituting (31) in (27),

$$L = \frac{(c - d)D - aW}{a + b} + W = KW$$

$$K = \frac{(c - d)D - aW + aW + bW}{(a + b)W}$$

Therefore,

$$K = \frac{(c - d)D + bW}{(a + b)W}$$

Example 2.—The following example involves all six of the equations of equilibrium. In Fig. 22, which is a diagrammatic sketch of a tripod landing gear, members BO and CO are pin-jointed at both ends. Member AOD is pin-jointed at O to member OB and OC, but is restrained at A by a joint designed to take torsion but no bending. B and C are pin joints. Find the reactions at A, B, and C due to the load P. (If any additional restraint is assumed, such as a fixed joint at A, B, C, or O, the structure will be statically indeterminate.) See Table X for direction cosines.

The components are:

$$\begin{array}{lll} T_x = 0.570T & B_x = 0.186B & C_x = 0.119C \\ T_y = 0.456T & B_y = 0.312B & C_y = 0.795C \\ T_z = 0.684T & B_z = 0.932B & C_z = 0.596C \end{array}$$

Let $P = 1$ lb.

The equations of equilibrium are:

$$\Sigma F_x = -A_x + 0.186\,B + 0.119\,C = 0 \tag{32}$$

$$\Sigma F_y = -A_y - 0.312\,B + 0.795\,C = 0 \tag{33}$$

$$\Sigma F_z = A_z - 0.932\,B - 0.596\,C + 1 = 0 \tag{34}$$

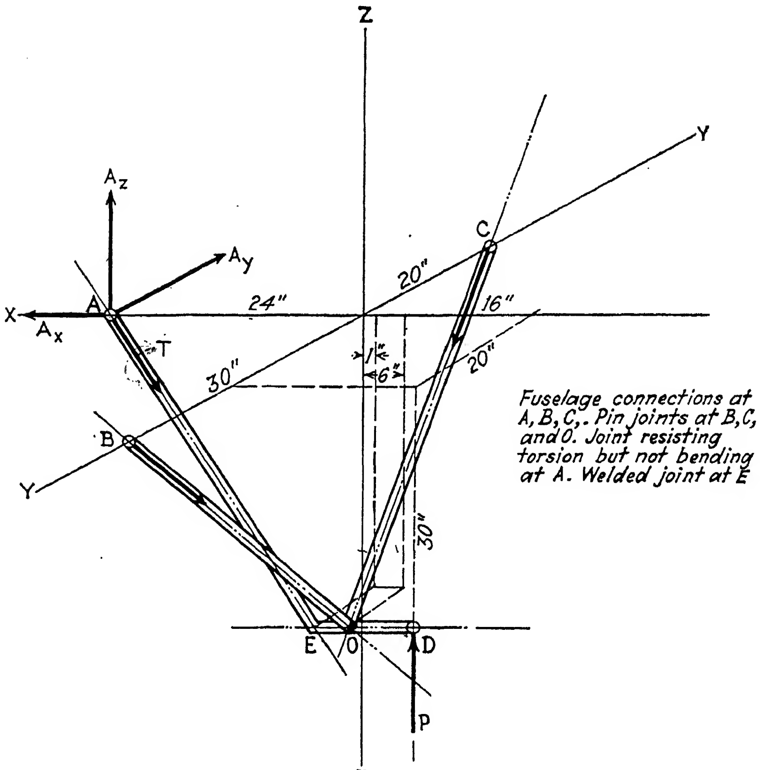


FIG. 22.—Free-body diagram of landing gear. Three-dimension problem in statics.

TABLE X.—CALCULATION OF DIRECTION COSINES

Point	Coordinates			Member	L	Direction cosines		
	X	Y	Z			α	β	γ
A	−24	0	0	AE	43.9	0.570	0.456	0.684
B	0	30	0	BO	32.2	0.186	0.312	0.932
C	0	−20	0	CO	50.3	0.119	0.795	0.596
D	16	20	−30	DO	10.0	1.000	0	0
E	1	20	−30	EO	5.0	1.000	0	0
O	6	20	−30					

$$\Sigma M_x = 0.570\ T - 30(0.932)\ B + 20(0.596)\ C + 20 = 0 \quad (35)$$
$$\Sigma M_y = 24A_z - 16 + 0.456\ T = 0 \quad (36)$$
$$\Sigma M_z = 24A_y - 30(0.186)\ B + 20(0.119)\ C - 0.684\ T = 0 \quad (37)$$

We solve as follows:

$$\begin{array}{llr}
-24 \times (34) & -24\ A_z + 22.4\ B + 14.3\ C = 24 & \\
(36) & 24\ A_z + 0.456\ T = 16 & \\
& & \\
\text{Add} & 0.456\ T + 22.4\ B + 14.3\ C = 40 & (38) \\
0.8 \times (35) & 0.456\ T - 22.4\ B + 9.5\ C = -16 & \\
& & \\
\text{Add} & 0.912\ T + 23.8\ C = 24 & (39) \\
(37) & 24\ A_y - 5.6\ B + 2.4\ C - 0.684\ T = 0 & (40) \\
24 \times (33) & -24\ A_y - 7.5\ B + 19.1\ C = 0 & \\
& & \\
\text{Add:} & -13.1\ B + 21.5\ C - 0.684\ T \quad 0 & (41) \\
1.71 \times (41) & -1.17\ T - 22.4\ B + 36.8\ C = 0 & \\
(38) & 0.456\ T + 22.4\ B + 14.3\ C = 40 & (42) \\
& & \\
\text{Add:} & -0.714\ T + 51.1\ C = 40.0 & (43) \\
0.783 \times (39) & 0.714\ T + 18.6\ C = 18.8 & \\
& & \\
\text{Add} & 69.7\ C = 58.8 & (44) \\
& C = 0.845 \text{ lb.} & \\
(43) & 0.714\ T = 51.1 \times 0.845 - 40 = 3.1 & (45) \\
& T = 4.34 \text{ lb.-in.} & \\
(42) & 22.4\ B = 40 - 14.3 \times 0.845 - 0.456 \times 4.34 & (46) \\
& B = 1.155 \text{ lb.} & \\
(32) & A_x = 0.186 \times 1.155 + 0.119 \times 0.845 & \\
& = 0.215 + 0.101 = 0.316 \text{ lb.} & (47) \\
(33) & A_y = 0.795 \times 0.845 - 0.312 \times 1.155 & \\
& = 0.672 - 0.361 = 0.311 \text{ lb.} & (48) \\
(34) & A_z = 0.932 \times 1.155 + 0.596 \times 0.845 - 1 & \\
& = 1.076 + 0.504 - 1 = 0.580 \text{ lb.} & (49)
\end{array}$$

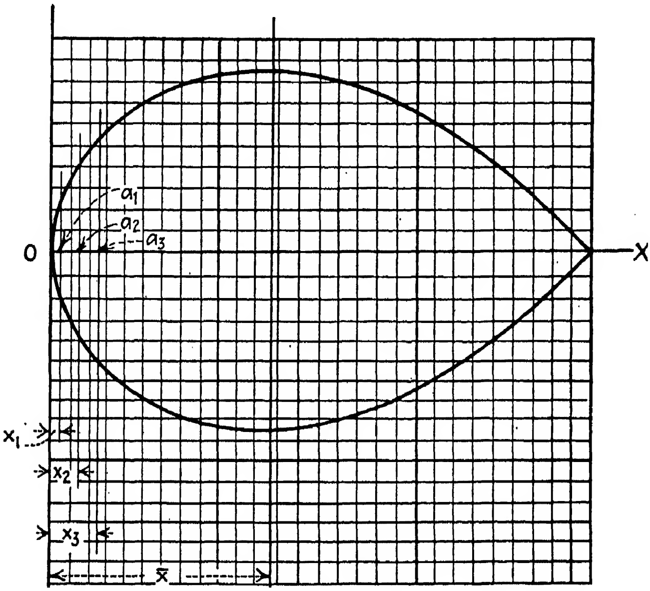
$$\begin{aligned}
A &= \sqrt{A_x^2 + A_y^2 + A_z^2} \\
&= \sqrt{0.3365 + 0.0937 + 0.0968} \\
&= \sqrt{0.5270} = 0.726 \text{ lb.}
\end{aligned}$$

See Sec. IV, Chap. XVIII, for a detailed consideration of landing-gear problems of this type.

CHAPTER VIII

PROPERTIES OF PLANE SECTIONS

106. Approximate Methods.—In aircraft stress analyses, it is frequently necessary to compute the *areas, centroids,* and *moments of inertia* of irregular cross sections and cross sections of uncon-



Area of each square = 0.1 × 0.1 = 0.01 sq. in.

FIG. 23.—Neutral axis and moment of inertia by approximate method.

ventional shapes such as the streamlined section. Probably the simplest course to follow in computing these quantities for irregular sections not subject to mathematical analyses is that indicated by the definition of the quantities. For example,

$$A = (a_1 + a_2 + a_3 + a_4 + \qquad + a_n) \tag{50}$$
$$= \sum_1^n a$$

in which A is the total areas and a_1, a_2, etc., are the areas of the parts taken as the basis of calculations. For example, in Fig. 23,

which illustrates the cross section of a strut, the total area is as indicated in equation (50), the value of each term of which may be computed.

If the section is too irregular for the calculation of the increments of area, the work of calculation may be facilitated by plotting the section on coordinate paper, laid off in squares of 0.1-in. sides. These unit squares then will be 0.1 in. by 0.1 in., or 0.01 sq. in. in area. For example, in Fig. 23 we have a streamlined section plotted on 0.1-in. cross-section paper. The area, by counting the squares, is in square inches,

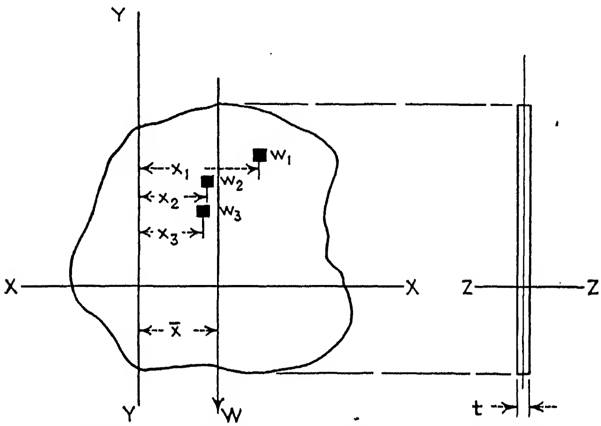


FIG. 24.—Location of center of gravity and centroid.

$$A = (a_1 + a_2 + a_3 + \cdots) = (0.04 + 0.08 + 0.10 + 0.12 + \cdots) \quad (51)$$

107. The Centroidal Axis.—In determining the centroid of the area, use is made of the principle of moments. For example, in Fig. 24,

$$W\bar{x} = x_1w_1 + x_2w_2 + x_3w_3 = \Sigma xw \quad (52)$$

or since

$$W = w_1 + w_2 + w_3 = \Sigma w \quad (53)$$

we have

$$\bar{x} = \frac{x_1w_1 + x_2w_2 + x_3w_3}{w_1 + w_2 + w_3} = \frac{\Sigma xw}{\Sigma w} \quad (54)$$

Now if the weights, w_1, w_2, w_3, etc., are the weights of a section of a thin plate of thickness t and density ρ per unit volume, then

$$w_1 = \rho t a_1, \qquad w_2 = \rho t a_2 \tag{55}$$

$$\bar{x} = \frac{x_1 \rho t a_1 + x_2 \rho t a_2 + x_3 \rho t a_3 + \cdots}{\rho t a_1 + \rho t a_2 + \rho t a_3 + \cdots} = \frac{\rho t \Sigma x a}{\rho t \Sigma a} \tag{56}$$

or for the areas,

$$\bar{x} = \frac{\Sigma x a}{\Sigma a} \tag{57}$$

which, written in integral form, is

$$x = \frac{\int x\,dA}{\int dA} \tag{58}$$

The use of 0.1-in. coordinate paper is also very convenient for the evaluation of equation (57) for an irregular section as illustrated in Fig. 23. Applying equation (57), we tabulate the calculation as in Table XI.

TABLE XI.—TABULATION OF CALCULATION FOR CENTROID

Area	Area, a, square inches	x, inches	xa
a_1			
a_2			
a_n			
	Σa		Σxa

108. Centroidal Axes of Small Areas.—For small areas which are usual in aircraft structures, it appears to be inadvisable to use graphical methods for finding either the centroid or the moment of inertia.

A very practical and accurate method of finding a centroidal axis of an area is as follows: Plot the area on stiff cardboard or a sheet of metal. Cut the section out. Now balance the cut-out section as nearly as possible over a wire of small diameter. The line of contact with the wire will be the centroidal axis. In finding the centroid of a cross section of a beam of very thin

sheet metal, for example, as in Fig. 25, a wire of small diameter may be bent to the proper shape for use in place of the cardboard section. The bent wire may then be balanced across another small straight wire to find $\bar{x}$.

109. Centroid by Integration. As an example of the determination of a centroidal axis of a section of thin sheet material,

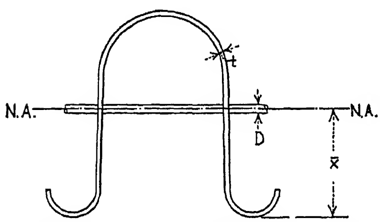


FIG. 25.—Neutral axis of thin metal spar by experiment.

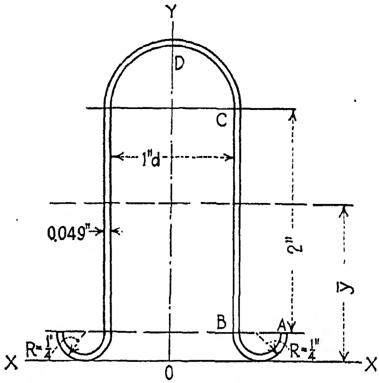


FIG. 26.—Centroid and moment of inertia of thin metal spar section by integration.

consider the section shown in Fig. 26. We are to find $\bar{y}$.

$$\bar{y} = \frac{\int y dw}{\int dw} = \frac{\int y\rho t ds}{\int \rho t ds} = \frac{\int y ds}{\int ds} \tag{59}$$

in which ρ is the density of the material and s is the distance along the neutral line of the thin sheet.

Supplying the limits in equation (59), we have,

$$\bar{y} = \frac{2\int_A^B y ds + 2\int_B^C y ds + 2\int_C^D y ds}{2\int_A^B ds + 2\int_B^C ds + 2\int_C^D ds} \tag{60}$$

$$\bar{y} = \frac{2\left[\int_A^B y ds + \int_B^C y ds + \int_C^D y ds\right]}{2[\frac{1}{4}\pi + 2 + \frac{1}{4}\pi]} \tag{61}$$

Since for a semicircle, as in Fig. 33,

$$\bar{y}_0 = \frac{\int y dw}{\int dw} = \frac{2\rho 0.049 r^2 \int_0^{\frac{\pi}{2}} \cos\theta d\theta}{2\rho 0.049 r^2 \int_0^{\frac{\pi}{2}} d\theta}$$

$$\frac{\left[\sin\theta\right]}{\left[\theta\right]_0^{\frac{\pi}{2}}} \qquad \frac{2r}{\pi} \tag{62}$$

We find for equation (61),

$$\bar{y} = \frac{\Sigma(Ly)}{T} = \frac{2(0.785)(0.250 - 0.159) + 2(2)(1.25) + (2.57)(1.57)}{2(0.785) + 4 + 1.57}$$

or

$$\bar{y} = \frac{9.18}{7.14} = 1.285 \text{ in.} \tag{63}$$

110. Moment of Inertia.—The definition of the moment of inertia of an area about an axis is

$$I_y = \int x^2 dA \tag{64}$$

It will be recalled that this integral is important because of its appearance in the modulus-of-rupture formula.

The integral may be determined approximately by the formula

$$I_x = (x_1^2 a_1 + x_2^2 a_2 + x_3^2 a_3 + \qquad) \tag{65}$$

For example, in Fig. 23, the moment of inertia of the streamlined section about the *oy* axis is given by the equation (65). The tabulation will be as in Table XII.

TABLE XII.—TABULATION OF CALCULATIONS FOR MOMENT OF INERTIA

Area, designation	Area, a, square inches	x, inches	x^2a
a_1			
a_2			
a_3			
	Σa		$\Sigma x^2 a$

In most cases, the moment of inertia about the centroidal axis is required. Knowing the area of the cross section and the location of the centroidal axis, it is a very simple matter to find the moment of inertia about any axis by the use of the parallel-axis theorem.

111. Parallel-axis Theorem.—Quite often the parallel-axis theorem is used erroneously. Let us note the assumptions in its derivation:

Let us assume in Fig. 27 that I_y is known. We are to find $I_{y'}$. We have by definition,

$$I_{y'} = \int (h + x)^2 dA \tag{66}$$

Or, upon expansion,

$$I_y = \int h^2 dA + 2\int hx dA + \int x^2 dA \tag{67}$$

Since

$$\int h^2 dA = h^2 \int dA = h^2 A \tag{68}$$

and

$$\int x^2 dA = I_y \tag{69}$$

we have

$$I_{y'} = I_y + h^2 A + 2h \int x dA \tag{70}$$

which is the general form of the parallel-axis theorem.

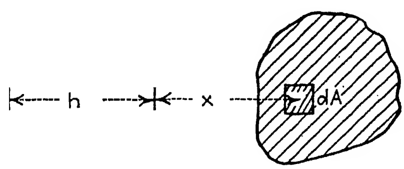


FIG. 27.—Parallel axis theorem.

Now if the oy axis were the centroidal axis as in Fig. 28, the integral $\int x dA$ would become zero. Hence, using a bar above the symbols to refer to the centroidal or gravity axis, we have the special case of the parallel-axis theorem. Writing the general symbol I for I_y, we have

$$I = \bar{I} + h^2 A \tag{71}$$

It should be especially noted that equation (71) *refers to the gravity axis of the section.*

112. Polar Moment of Inertia.—In problems in torsion it is necessary to know the polar moment of inertia of a section. Letting J be the symbol for the polar moment of inertia, we have

$$\int r^2 dA \tag{72}$$

We note in Fig. 29 that

$$r^2 = x^2 + y^2 \tag{73}$$

Hence

$$J = \int (x^2 + y^2)dA = \int x^2 dA + \int y^2 dA \tag{74}$$

or

$$J = I_y + I_x \tag{75}$$

In certain symmetrical areas, such as that of a cylinder, I_y and I_x are equal, so that

$$J = 2I \qquad \text{or} \qquad I \quad J \tag{76}$$

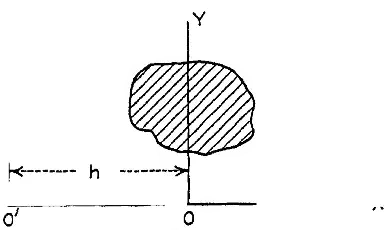


FIG. 28.—Moment of inertia about parallel axis through center of gravity.

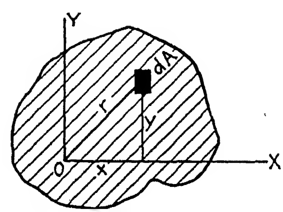


FIG. 29.—Polar moment of inertia.

This relation is sometimes useful in the simplification of the calculation of I. For example, for the circular cylinder,

$$J = \int r^2 dA = \int^{\pi} \int_0^a r^2 (r d\theta) dr \tag{77}$$

Evaluating the integral,

$$J = \quad 2\pi a^4 \qquad \pi a^4 \tag{78}$$

Since

$$I = \frac{J}{2}$$

$$I = \frac{\pi a^4}{4} \tag{79}$$

113. Moment of Inertia of the Cross-sectional Area of a Streamlined Surface.—For the purpose of this illustration, let us assume the streamlined section is composed of an elliptical nose

and a parabolic tail, as illustrated in Fig. 30. We are to find the moment of inertia about the x-x axis. (The example is for illustration of the *method.*) If we let I_e be the moment of inertia of the elliptical nose, and I_p the moment of inertia of one side of

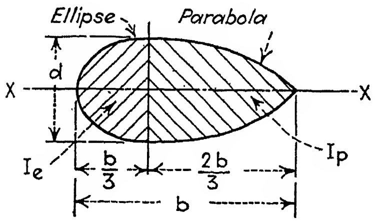


FIG. 30.—Streamlined section of elliptic nose and parabolic tail.

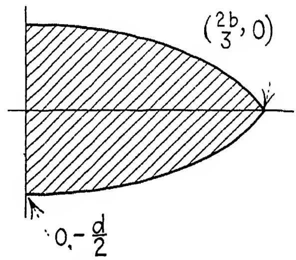


FIG. 31.—Parabolic tail of streamlined section.

the parabolic tail, we have, letting $2a$ and $2b$ be the x and y axes respectively of the ellipses,

$$I_e = \int y^2 dA = 2\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} \int_0 y^2 dx dy$$

$$= \frac{2a}{b}\int_0^b y^2(\sqrt{b^2 - y^2})dy \tag{80}$$

If we let

$$y = b \sin \theta, \qquad dy = b \cos t$$

and

Thus

$$I_e = 2ab^2 \int \sin^2 \theta \cos^2 d\theta = \frac{ab^3}{2}\int \sin^2 2\theta d\theta$$

or

$$\frac{ab^3}{4}\int (1 - \cos 4\theta) d\theta = \frac{ab^3}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

so that

$$I_e = \pi ab^3 \tag{81}$$

We note for future reference that the moment of inertia of the complete ellipse is

$$I_e = \pi a b^3 \tag{82}$$

The equation of a parabola symmetrical with respect to the y-axis is

$$x^2 = 4p(y - k) \tag{83}$$

We note from Fig. 31 that when $y = 0$, $x = 2b/3$, and when $x = 0$, $y = -d/2$. Substituting these limits in turn in equation (83), we have

$$0 \doteq 4p\left(-\frac{d}{2} - k\right)$$

from which

$$k = -d/2$$

and

$$4/9\, b^2 = 4p(-k)$$

From this,

$$p = +\frac{2}{9}\frac{b^2}{d} \tag{84}$$

from which,

$$x = \frac{2b}{3}\sqrt{\frac{2}{d}\left(y + \frac{d}{2}\right)} \tag{85}$$

We therefore have

$$I_p = 2\int_0^{-\frac{d}{2}}\int_0^x y^2 dx dy \tag{86}$$

or, integrating with respect to x,

$$I_p = \frac{4b}{3}\sqrt{\frac{2}{d}}\int_{\frac{d}{2}}^{0} y^2\left(\sqrt{y + \frac{d}{2}}\right)dy \tag{87}$$

To integrate, let

$$y + \frac{d}{2} = z^2, \qquad y = z^2 - \frac{d}{2}$$

and

$$dy = 2z dz$$

Thus

$$\int y^2\left(\sqrt{y + \frac{d}{2}}\right)dy = 2\int\left(z^2 - \frac{d}{2}\right)^2 z^2,$$

$$= 2\int\left(z^6 - dz^4 + \frac{d^2z^2}{4}\right)dz = 2\left[\frac{z^7}{7} - \frac{dz^5}{5} + \frac{d^2z^3}{12}\right]$$

$$= 2\ \frac{\left(y + \frac{d}{2}\right)^{7/2}}{7} - \frac{d\left(y + \frac{d}{2}\right)^{5/2}}{\quad} + \frac{d^2\left(y + \frac{d}{2}\right)^{3}}{\quad}$$

$$= 2\ \frac{d^{7/2}}{7 \times 8\sqrt{2}} \quad \frac{d^{7/2}}{5 \times 4\sqrt{2}} + \frac{d^{7/2}}{24\sqrt{2}}\Bigg] = \frac{2d^3\sqrt{d}}{4\sqrt{2}}\left[\frac{1}{14} - \frac{1}{5} + \right. \tag{88}$$

$$I_p = \frac{2 \times 4bd^3}{315} \quad \frac{bd^3}{39.4} \tag{89}$$

Since

$$I_e = \frac{\pi ab^3}{\quad} \qquad a = \frac{1}{3}b, \qquad \text{and} \qquad b = \frac{d}{2}$$

$$I_e = \frac{\pi bd^3}{3 \times 8 \times 8} = \frac{bd^3}{61.1} \tag{90}$$

Thus

$$I = \frac{bd^3}{61.1} + \frac{bd^3}{39.4} \quad \frac{bd^3}{24} \tag{91}$$

114. Ellipse of Inertia.—It sometimes is desirable to find the moment of inertia about an inclined axis. For this purpose, the

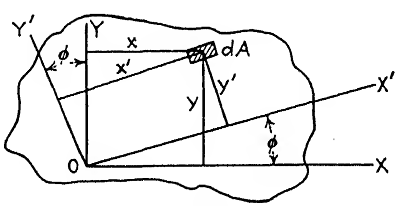


Fig. 32.—Moment of inertia about inclined axis.

ellipse of inertia is useful. Let us determine such an ellipse and note its application. We write for Fig. 32,

$$I_{x'} = \int (y')^2 dA = \int (y \cos \phi - x \sin \phi)^2 dA \tag{92}$$

$$= \cos^2 \phi \int y^2 dA + \sin^2 \phi \int x^2 dA - 2 \sin \phi \cos \phi \int xy dA \tag{93}$$

$$I_{x'} = I_x \cos^2 \phi + I_y \sin^2 \phi - 2k \cos \phi \sin \phi \tag{94}$$

in which k is the product of inertia defined by $\int xy dA$.

Likewise we find

$$I_{y'} = I_x \sin^2 \phi + I_y \cos^2 \phi + 2k \cos \phi \sin \phi \tag{95}$$

We note in passing, by adding equations (94) and (95), that

$$I_{x'} + I_{y'} = I_x + I_y \tag{96}$$

This is apparent also, because each side of the equation, as previously noted, is the polar moment of inertia of the section.

Now in order to represent equation (94) as an ellipse of inertia, let us assume

$$Me^4 = I_x x^2 + I_y y^2 - 2kxy \tag{97}$$

which is the equation of an ellipse in which M is a mass and e is any linear length. This choice is necessary if we are to keep the dimension of the two sides of the equation homogeneous with respect to the units. Let ρ be the polar radius to any point on the curve of the ellipse. Then if ϕ is the angle between this radius and the x-axis, we have

$$y = \rho \sin \phi \qquad \text{and} \qquad x = \rho \cos \phi \tag{98}$$

Thus equation (97) becomes

$$\frac{Me^4}{\rho^2} = I_x \cos^2 \phi + I_y \sin^2 \phi - 2k \cos \phi \sin \phi \tag{99}$$

Thus we have

$$\frac{Me^4}{\rho^2} \tag{100}$$

From equation (100) it may be concluded that the moment of inertia about any axis is inversely proportional to the square of the polar radius of the ellipse. Thus if we draw the ellipse

$$+ \frac{y}{b^2} \tag{101}$$

so that

$$a = \frac{}{\sqrt{I_x}} \qquad \text{and} \qquad b = \frac{}{\sqrt{I_y}} \tag{102}$$

then

$$\sqrt{I_x}$$

We thus have

$$\frac{a}{\rho} = \frac{1/\sqrt{I_x}}{1/\sqrt{I_{x'}}}, \qquad \frac{a^2}{\rho^2} = \frac{I_{x'}}{I_x}$$

or

$$I_{x'} = I_x\frac{a^2}{\rho^2} \tag{103}$$

The principal axes, are, it may well be recalled, perpendicular to each other. One is a maximum with respect to ϕ, and the other is a minimum. Let us find ϕ for the location of the principal axes. Thus

$$\frac{dI_{x'}}{d\phi} = \sin 2\phi(I_y - I_x) - 2k\cos 2\phi = 0$$

from which

$$\tan\phi = \frac{2k}{I_y - I_x} \tag{104}$$

115. Moment of Inertia of a Cross Section of Thin Metal.—In structures of thin sheet metal, full monocoque, or semi-monocoque, it becomes necessary at times to compute the moments of inertia of cross-sectional areas. For example, let us compute the moment of inertia of the cross-sectional area of the tube shown in Fig. 33. Let us assume that the ratio of radius to thickness is very large so that the radius may be taken as the radius of the neutral line of the surface. We have, therefore,

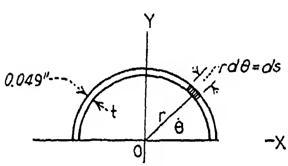


FIG. 33.—Moment of inertia of thin walled tube by integration.

$$= \frac{J}{2} = \frac{1}{2}\int r^2 dA = \frac{1}{2}r^2\int_0 rtd\theta$$
$$= \tfrac{1}{2}r^3t(2\pi) = \pi r^3 t \tag{105}$$

Let us, in order to illustrate a general method, which we will use to advantage later, find I_x by another method. We have for a solid cylinder,

$$I = \frac{\pi r^4}{4} \tag{106}$$

From this

$$\frac{dI}{dr} = \pi r^3 \qquad \text{and} \qquad dI = \pi r^3 dr \tag{107}$$

If we write I_x for dI, and t for dr, we have

$$I_x = \pi r^3 t, \tag{108}$$

as before.

Let us apply this method to an elliptical section, such as a monocoque fuselage. We have for the solid section, as indicated in equation (82),

$$I = {}^{\pi a^3 b} \tag{109}$$

$$dI = I_y = \frac{3\pi a^2 b}{4} da + \frac{\pi a^3}{4} db \tag{110}$$

Write t for da and db, and we have

$$I_y = \frac{3\pi a^2 bt}{4} + \frac{a^3}{4} t \tag{111}$$

$$= \tfrac{1}{4}\pi a^2 (3b + a)t \tag{112}$$

Let us now apply this method to the formula for the solid streamlined section and obtain a formula for the streamlined tube. We have

$$I = \frac{bd^3}{24} \text{ [see equation (91)]} \tag{113}$$

$$dI = \frac{d^3}{24} db + \frac{3bd^2}{24} dd \tag{114}$$

or since d and b are taken as the full width and length of the section, $db = 2t$, and $dd = 2t$, hence

$$I_x = \frac{2d^3 t}{24} + 2\frac{3bd^2 t}{24} = \frac{2d^2}{24}(d + 3b)t \tag{115}$$

in which d and b have the dimensions as indicated in Fig. 34. The area of the streamlined section of Fig. 35, as the student may easily verify, is

$$0.785bd \tag{116}$$

Let us apply the derivative method to the area formula and derive a formula for computing the area of the cross section of a streamlined tube. Thus

$$dA = (0.785d)db + (0.785b)dd \tag{117}$$

or

$$A_t = 0.785(b + d)(2t) \tag{118}$$

so that

$$A_t = 1.57(b + d)t \tag{119}$$

116. Moment of Inertia of an Airfoil Section.—We note in Fig. 35 that, if the symmetrical streamlined section be split

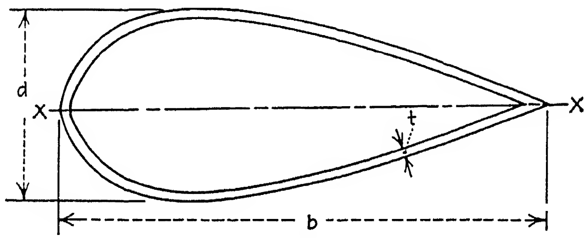


FIG. 34.—Approximate moment of inertia of a streamlined tube.

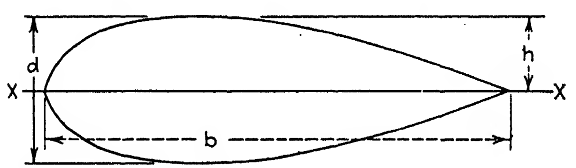


FIG. 35.—Solid streamlined section.

along the line x-x, two airfoil sections will be formed. We have for the entire section,

$$I = \frac{bd^3}{24} = \frac{b(2h)^3}{24} = \frac{bh^3}{3} \tag{120}$$

The moment of inertia of the upper half of this section about the x-x axis is

$$\frac{1}{2}\left(\frac{bh^3}{3}\right) = \frac{bh^3}{6} \tag{121}$$

The moment of inertia about the centroidal axis x'-x' (Fig. 36) is by the parallel-axis theorem,

$$\bar{I} = I_a - (\bar{y})^2A \tag{122}$$

in which A is slightly less than one-half the area of the symmetrical section, that is,

$$A = \frac{0.785}{2}bd = 0.78bh \text{ (approx.)} \tag{123}$$

In general $\bar{y}$ is approximately 40 per cent of the maximum ordinate, which may be verified in each case by an experiment previously described. Thus

$$\begin{aligned}(\bar{y})^2A &= (0.4h)^2(0.78bh)\\ &= \tfrac{1}{8}bh^3 \text{ (approx.)}\end{aligned} \tag{124}$$

Hence,

$$\bar{I} = \frac{bh^3}{6} - \frac{1}{8}bh^3$$

or

$$\bar{I} = \frac{bh^3}{24} \tag{125}$$

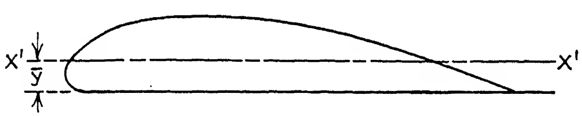


FIG. 36.—Section of propeller blade.

117. Area of Cross Section of Wing Shell.—The shell, in this case, is a tube in the form of an airfoil section. As noted in equation (123), the area of the solid cross section is

$$A = 0.78bh \tag{126}$$

From which

$$dA = (0.78h)db + (0.78b)dh \tag{127}$$

So that for the shell,

$$\begin{aligned}A &= 0.78(h + b)(2t)\\ &= 1.56(h + b)t\end{aligned} \tag{128}$$

118. The Length of the Periphery of an Airfoil.—We note in passing that the area of the cross section of the shell wing is the length of the periphery times the thickness; that is

$$A = ct \tag{129}$$

in which c is the length of the periphery. Equating equation (129) to (128), we find

$$c = 1.56(h + b) \tag{130}$$

119. Other Formulas.—Air Corps Information Circular 597 gives the following data on airfoil sections suitable for propeller blades (see Fig. 37):

$$A = 0.724bh \tag{131}$$
$$I_{xx} = 0.0454bh^3 \tag{132}$$
$$I_{yy} = 0.0418b^3h \tag{133}$$

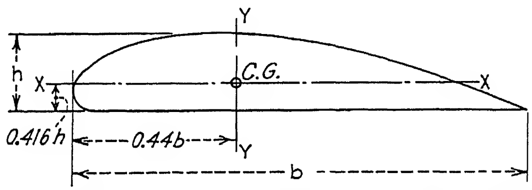


FIG. 37.—Moment of inertia and neutral axis of solid airfoil section.

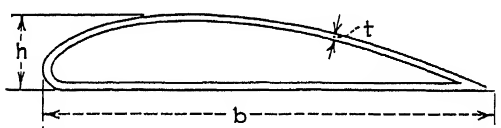


FIG. 38.—Shell section of b-chord and h-ordinate.

From these data, we determine the area and moments of inertia for a tubular section (Fig. 38) as follows: From equation (131),

$$dA = 0.724bdh + 0.724hdb \tag{134}$$

From this, writing dh and db as $2t$,

$$\begin{aligned} A &= 0.724(b + h)(2t) \\ &= 1.448(b + h)t \end{aligned} \tag{135}$$

Likewise

$$I_{xx} = 0.0908h^2(h + 3b)t \tag{136}$$

From equation (125) we obtain a similar formula,

$$I_{xx} = \frac{1}{12}h^2(h + 3b)t = 0.084h^2(h + 3b)t \tag{137}$$

It will be noted that the two equations differ by only 6 per cent.

TABLE XIII.—PROPERTIES OF ROUND TUBE SECTIONS

Diameter, in.	Thickness, in.	Gage, B and W	Area, sq. in.	Round		
				I, in.^{4}	ρ, in.	$\frac{I}{y}$ in.^{3}
¼	0.022	24	0.01576	0.000103	0.08098	0.000827
	0.028	22	0.01953	0.0001222	0.07911	0.000978
	0.035	20	0.02364	0.001402	0.07701	0.001122
⅜	0.022	24	0.02440	0.000381	0.1251	0.002035
	0.028	22	0.03053	0.0004600	0.1231	0.002463
	0.035	20	0.03739	0.0005459	0.1208	0.002912
	0.049	18	0.05018	0.0006817	0.1166	0.003636
	0.058	17	0.06776	0.0007498	0.1139	0.003999
½	0.035	20	0.05113	0.0013898	0.1649	0.005559
	0.049	18	0.06943	0.0017860	0.1604	0.007144
	0.058	17	0.08054	0.002001	0.1576	0.008003
⅝	0.035	20	0.06487	0.002833	0.2090	0.009065
	0.049	18	0.08867	0.003704	0.2044	0.01185
	0.058	17	0.10331	0.004195	0.2015	0.01342
¾	0.035	20	0.07862	0.005036	0.2531	0.01343
	0.049	18	0.10791	0.006661	0.2484	0.01776
	0.058	17	0.12609	0.007601	0.2455	0.02027
⅞	0.035	20	0.09236	0.008161	0.2972	0.01865
	0.049	18	0.12715	0.010882	0.2926	0.02487
	0.058	17	0.14887	0.012484	0.2896	0.02853
1	0.035	20	0.10611	0.012368	0.3414	0.02474
	0.049	18	0.14640	0.016594	0.3367	0.03319
	0.058	17	0.17164	0.019111	0.3337	0.03822
	0.065	16	0.19093	0.02097	0.3314	0.04193
1⅛	0.035	20	0.11985	0.01782	0.3856	0.03168
	0.049	18	0.16564	0.02402	0.3808	0.04270
	0.058	17	0.19442	0.02775	0.3778	0.04933
	0.065	16	0.2165	0.03052	0.3755	0.05425
1¼	0.035	20	0.13360	0.02467	0.4298	0.03948
	0.049	18	0.18488	0.03339	0.4250	0.05342
	0.058	17	0.2172	0.03867	0.4219	0.06187
	0.065	16	0.2420	0.04260	0.4196	0.06816
1⅜	0.035	20	0.14734	0.03309	0.4739	0.04814
	0.049	18	0.2041	0.04492	0.4691	0.06534
	0.058	17	0.2400	0.05213	0.4661	0.07583
	0.065	16	0.2675	0.05753	0.4637	0.08367
	0.083	14	0.3369	0.07059	0.4577	0.1027
1½	0.035	20	0.16109	0.04324	0.5181	0.05765
	0.049	18	0.2234	0.05885	0.5133	0.07847
	0.058	17	0.2628	0.06841	0.5102	0.09121
	0.065	16	0.2930	0.07558	0.5079	0.1008
	0.083	14	0.3695	0.09305	0.5018	0.1241
	0.095	13	0.4193	0.10394	0.4979	0.1386
	0.120	11	0.5202	0.12478	0.4898	0.1664
1⅝	0.035	20	0.1748	0.05528	0.5623	0.06803
	0.049	18	0.2426	0.07540	0.5575	0.09279
	0.058	17	0.2855	0.08776	0.5544	0.1080
	0.065	16	0.3186	0.09707	0.5520	0.1195
	0.083	14	0.4021	0.11985	0.5459	0.1475
	0.095	13	0.4566	0.13413	0.5420	0.1651
	0.120	11	0.5674	0.16166	0.5338	0.1990

TABLE XIII.—PROPERTIES OF ROUND TUBE SECTIONS.—(*Continued*)

Diameter, in.	Thickness, in.	Gage, B and W	Area, sq. in.	Round		
				I, in.^{4}	ρ, in.	$\frac{I}{y}$, in.^{3}
1¾	0.035	20	0.1886	0.06936	0.6064	0.07927
	0.049	18	0.2618	0.09478	0.6016	0.1083
	0.058	17	0.3083	0.11046	0.5986	0.1262
	0.065	16	0.3441	0.12230	0.5962	0.1398
	0.083	14	0.4347	0.15136	0.5901	0.1730
	0.095	13	0.4939	0.16967	0.5861	0.1939
	0.120	11	0.6145	0.2052	0.5779	0.2345
1⅞	0.035	20	0.2023	0.08565	0.6507	0.09136
	0.049	18	0.2811	0.1194	0.6527	0.1274
	0.058	17	0.3311	0.13677	0.6427	0.1459
	0.065	16	0.3696	0.15156	0.6403	0.1617
	0.083	14	0.4673	0.18797	0.6342	0.2005
	0.095	13	0.5312	0.2110	0.6302	0.2251
	0.120	11	0.6616	0.2559	0.6219	0.2727
2	0.035	20	0.2161	0.1043	0.6948	0.1043
	0.049	18	0.3003	0.1430	0.6901	0.1430
	0.058	17	0.3539	0.16696	0.6869	0.1670
	0.065	16	0.3951	0.18514	0.6845	0.1851
	0.083	14	0.4999	0.2300	0.6784	0.2301
	0.095	13	0.5685	0.2586	0.6744	0.2586
	0.120	11	0.7087	0.3144	0.6660	0.3144
2¼	0.305	20	0.2436	0.1494	0.7831	0.1328
	0.049	18	0.3388	0.2052	0.7782	0.1825
	0.058	17	0.3994	0.2401	0.7753	0.2134
	0.065	16	0.4462	0.2665	0.7729	0.2369
	0.083	14	0.5651	0.3322	0.7667	0.2953
	0.095	13	0.6432	0.3741	0.7627	0.3325
	0.120	11	0.8030	0.4568	0.7543	0.4060
2½	0.035	20	0.2710	0.2059	0.8716	0.1647
	0.049	18	0.3773	0.2834	0.8667	0.2267
	0.058	17	0.4450	0.3318	0.8636	0.2655
	0.065	16	0.4972	0.3688	0.8612	0.2950
	0.083	14	0.6302	0.4607	0.8550	0.3686
	0.095	13	0.7178	0.5197	0.8510	0.4158
	0.120	11	0.8972	0.6369	0.8425	0.5095
2¾	0.035	20	0.2985	0.2751	0.9600	0.2001
	0.049	18	0.4158	0.3793	0.9551	0.2759
	0.058	17	0.4905	0.4446	0.9520	0.3233
	0.065	16	0.5483	0.4944	0.9496	0.3596
	0.083	14	0.6954	0.6189	0.9434	0.4501
	0.095	13	0.7924	0.6991	0.9393	0.5084
	0.120	11	0.9915	0.8590	0.9308	0.6248
3	0.035	20	0.3260	0.3583	1.0484	0.2389
	0.049	18	0.4543	0.4946	1.0435	0.3298
	0.058	17	0.5361	0.5802	1.0404	0.3868
	0.065	16	0.5993	0.6457	1.0379	0.4305
	0.083	14	0.7606	0.8097	1.0317	0.5398
	0.095	13	0.8670	0.9156	1.0276	0.6104
	0.120	11	1.0857	1.1276	1.0191	0.7518
4	0.035	20	0.4360	0.8568	1.4018	0.4284
	0.049	18	0.6082	1.1870	1.3970	0.5935
	0.058	17	0.7183	1.3955	1.3939	0.6978
	0.065	16	0.8035	1.5557	1.3914	0.7779
	0.083	14	1.0214	1.9597	1.3852	0.9799
	0.095	13	1.1655	2.2228	1.3810	1.1114
	0.120	11	1.4627	2.7552	1.3724	1.3776

TABLE XIV.—PROPERTIES OF SECTIONS. ALUMINUM-ALLOY STREAMLINE TUBES[1]

Standard Fineness Ratio = 2.36

T.M. Sk. No.	Major axis, in.	Minor axis, in.	Wall thickness, in.	Area, sq. in.	Least moment of inertia, in.^{4}	Least radius of gyration, in.
61	6.474	2.760	0.134	1.984	1.967	0.996
59	5.428	2.300	0.109	1.332	0.934	0.837
60	5.428	2.300	0.148	1.784	1.215	0.825
164	5.057	2.143	0.109	1.255	0.749	0.773
46	4.720	2.000	0.095	1.023	0.535	0.723
41	4.389	1.860	0.083	0.823	0.379	0.678
73	4.389	1.860	0.095	0.934	0.427	0.676
162	4.050	1.716	0.042	0.407	0.158	0.623
161	4.050	1.716	0.065	0.065	0.237	0.626
42	4.050	1.716	0.083	0.731	0.295	0.635
294	3.977	1.694	0.138	1.244	0.431	0.589
43	3.710	1.572	0.065	0.533	0.181	0.582
72	3.710	1.572	0.072	0.612	0.198	0.569
159	3.710	1.572	0.083	0.729	0.224	0.554
214	3.375	1.430	0.049	0.380	0.104	0.523
215	3.375	1.430	0.057	0.436	0.120	0.525
44	3.375	1.430	0.065	0.477	0.135	0.531
160	3.375	1.430	0.072	0.552	0.147	0.516
105	2.700	1.144	0.042	0.254	0.046	0.423
45	2.700	1.144	0.058	0.360	0.061	0.410
171	2.703	1.140	0.065	0.398	0.066	0.407
163	2.528	1.071	0.065	0.382	0.054	0.376
106	2.360	1.000	0.049	0.294	0.034	0.340
263	2.191	0.928	1.049	0.251	0.027	0.328
170	2.063	0.876	0.049	0.231	0.023	0.316
93	1.874	0.781	0.035	0.163	0.012	0.271
246	1.874	0.781	0.049	0.214	0.016	0.273

[1] Data taken from Aluminum Company of America, blueprint dated Jan. 13, 1932.

CHAPTER IX

STATICALLY INDETERMINATE STRUCTURES

120. Definitions.—A *statically indeterminate structure* is a structure in which the stresses cannot be determined by the methods of statics alone. The outstanding characteristics of the two types of structures, statically determinate and statically indeterminate, are listed in Table XV.

TABLE XV.—CHARACTERISTICS OF STRUCTURES

Statically Determinate	Statically Indeterminate
1. Assumed rigid structure.	1. Elastic structure.
2. Has enough members, and only enough, to prevent the structure from collapsing.	2. Has more than enough members to prevent the structure from collapsing.
3. Contains no initial stresses.	3. If one member is too long or too short, initial stresses will be introduced.
4. Stresses in structure not affected by temperature.	4. Stresses affected by temperature.
5. Loads in members not a function of the characteristics of the members.	5. Loads in the members are functions of the characteristics of members, such as length, cross-sectional area, modulus of elasticity, etc.

The number of restraints to be removed in a statically indeterminate structure to make it statically determinate is the *degree of redundancy*. The restraining members removed are the *redundant members*. For example, consider the structure in Fig. 39*a*. The structure will collapse under any system of loading. However, if we insert another member, as *BD* in Fig. 39*b*, the structure is then rigid and will not collapse under an assigned load. If we insert yet another member, *AC* as in Fig. 39*c*, we then have more than enough members in the structure to prevent it from collapsing. The additional member in this case is a *redundant member*. The *degree of redundancy* is one.

121. Proportionality of Stress and Strain.—Methods of analyzing statically indeterminate structures are based, in all

cases, on the law of proportionality between stress and strain, that is, Hooke's law. This law applies equally as well to a

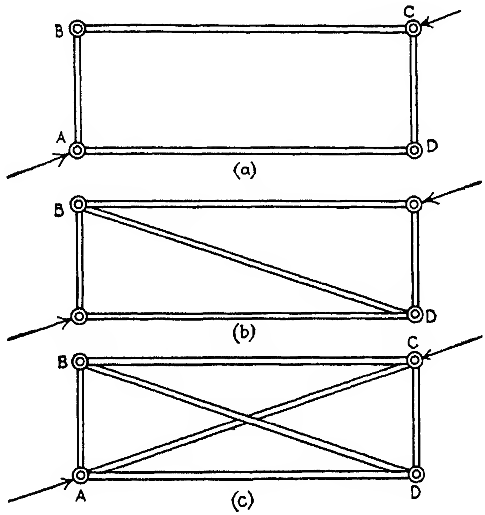


FIG. 39.—(a) Unstable; (b) statically determinate; (c) statically indeterminate structures.

structure of elastic material as to a slender rod. For example, suppose that we consider any structure such as the engine mount

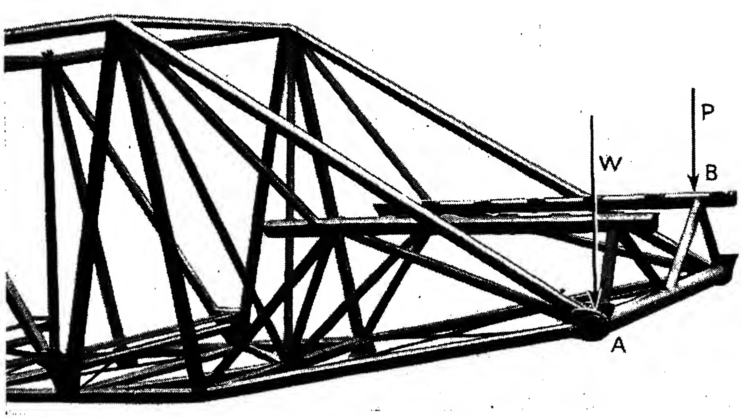


FIG. 40.—Effect of one load on structure independent of effect of another.

in Fig. 40. The deflection of point A in the direction of the load W is proportional to the load W regardless of the configura-

tion of the members of the structure. Also, the effect of the load W is independent of the load P applied at the joint B; that is, whether or not P is applied to the structure, the deflection of A caused by W is of the same magnitude. Also *the stress due to the load W in any member of the statically indeterminate structure, as in a statically determinate structure, is independent of the stress due to the load P. The stress in any member of the structure due to both load P and load W is the algebraic sum of the stresses induced by each load separately.* We should note, however, that this is true only in so far as the structure does not change its characteristics, as, for instance, by the elimination of a member. For example, suppose one of the members is a wire under an initial stress. If this wire becomes slack, the characteristic of the structure is then changed, and our law no longer holds at this point. The law holds for the new structure and for the structure unchanged, yet it does not hold *through* the change between the two. These laws hold regardless of whether the joints are pinned or fixed rigidly.

It should be noted that, inasmuch as the law of proportionality of stress and strain is the basis for the theoretical determination of stresses in a statically indeterminate structure, *methods of such analyses do not apply above the proportional limit of a material.*

122. Initial Stresses.—We have noted that, in the case of a statically indeterminate structure, the insertion of a redundant member, as shown in Fig. 39*c*, introduces initial stress unless the member is exactly the correct length, which is rarely the case. We should note, also, that if the member is of the correct length and is inserted properly, the change in temperature may induce a temperature stress due to the difference in the expansion of the members.

Methods of determining the stresses in a statically indeterminate structure are based on *the assumption that the initial stresses in the members are all zero. The total stress is the algebraic sum of the initial stress and the stress in the member due to the applied load.*

In some cases this initial stress affects the ultimate strength of the structure. In other cases it does not. The question is often asked: Why should we analyze the airplane structure for stress when a mechanic may tighten one of the bolts too tight, putting extra stresses in the structure which are not considered in our analysis? This problem requires investigation to find out

under what conditions the initial stresses are influential in the ultimate strength, and under what conditions they are not. Let us consider the structure in Fig. 41. Assume pin joints at A, B, and C, 45-degree angles at A and B, and a 90-degree angle at C. Apply the load W at point B. Let W equal 100 lb. The load in the member AB will be $100\sqrt{2}$. The load in the member CB will be 100 lb. Under the load of 100 lb., B will be deflected downward a small distance depending upon the size, the modulus of elasticity, and the other characteristics of the members of the structure. Now suppose that to this deformed condition we add

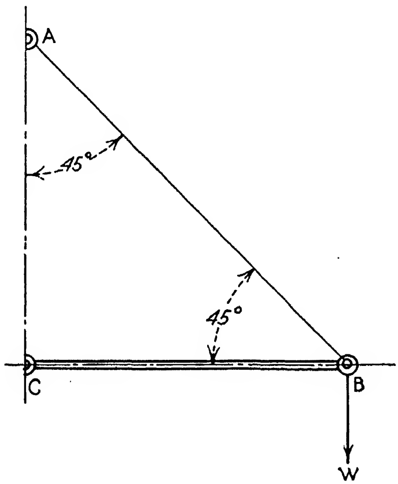


FIG. 41.

another member $A'B$ (see Fig. 42), similar in every way to the member AB. We add this member with the initial load of zero. Now, suppose we remove the 100-lb. weight and allow a redistribution of stress between the members. We note that since $A'B$ and AB are similar in every way, neglecting the small change in shape of the structure due to the deflection, the load will be distributed equally between these two members. That is, the stress in AB will be $50\sqrt{2}$ and the stress in $A'B$ will also be $50\sqrt{2}$. *We now have a statically indeterminate structure with known initial stresses.*

Apply successively increasing loads to the point B, such as 25 lb., then 100 lb.; then plot the deflection of B for each case, the load being the ordinate and the deflection being the abscissa

(see Fig. 43). The deflection will be proportional to the strain up to the point A where W is 100 lb. At this point A the member $A'B$ becomes slack again and the characteristic of our structure is changed. Our load-deflection curve will eventually lose its proportionality and drop to the point C where rupture occurs. It is quite obvious in this case *that failure in the structure is independent of the member $A'B$; consequently it is independent of the initial stress in the members.* We should note, however, that, if the initial stress in the member $A'B$ is great enough so that the point A occurs between B and C, then the initial stress does affect the ultimate strength. This rarely occurs in a structure. A general rule to follow in this case is that the initial stress of a wire member should not be over one-half the elastic limit of the material.

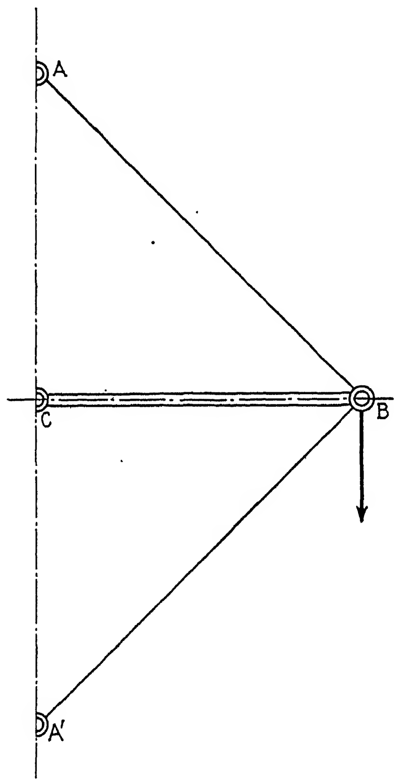


Fig. 42.

To continue the illustration, suppose that at the point B on the load-deflection curve $OABC$ we begin to relieve the load. Assume that while the member $A'B$ is still slack we remove the member from the structure. We then continue to relieve the load. Our load-deflection curve will be BAO'. $O'O$ then is the initial deflection of B due to the initial stress in the members of the structure. It will be noticed from this diagram that the *statically indeterminate structure was more rigid than the statically determinate structure,* since, with the member $A'B$ removed, the deformation is greater per unit of applied load. We may notice here, too, that some *structures under initial stress are more rigid than the same structures under no stress.* This principle is quite often made use of where a rigid structure of a light weight is required.

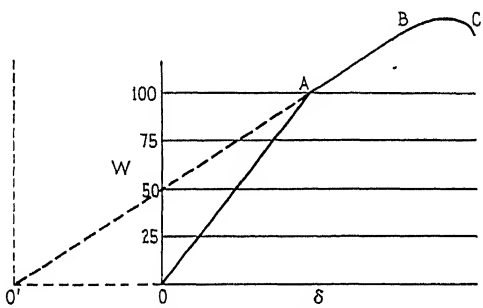


FIG. 43.—Load-deflection curve.

Let us consider another case quite similar in nature but different in results. Suppose that the members in this structure (Fig. 42) are all steel tubes. We note, therefore, that when the load in the member $A'B$ is relieved, the member $A'B$ does not drop out from the structure—that is, it still contributes to the strength of the structure. Let us draw a load-deflection curve for this condition (see Fig. 44). The curve continues in a straight line until the elastic limit of one member is reached or until the limit of *elastic stability* of $A'B$, as a strut, is attained. Now, the stress in the member AB continually increases. The stress in the member $A'B$ decreases until the load of 100 lb. is reached. It then increases in magnitude but with opposite sign. If the two members are similar, then the stress in the member AB will always be higher than the stress in the member $A'B$. We should note in this case, however, that the member AB would

fail in tension, and that the member $A'B$ would fail in compression from the standpoint of elastic instability; that is, it would fail as a strut.

123. Strength Increased by Initial Stress.—In certain cases the initial stress may be used to increase the ultimate strength of a structure. For example, considering Fig. 42 again, let us assume that the members AB and $A'B$ are steel tubes. Assume that the strength in tension of AB is $2{,}000\sqrt{2}$ lb. and the strength in compression (as a strut) of $A'B$ is $1{,}000\sqrt{2}$ lb. With reference to Fig. 44, we note that, with the member $A'B$ removed, the W-δ curve will be $O'ABC$, the maximum W occurring at B. Now with $A'B$ in place, and with no initial stress, the W-δ curve

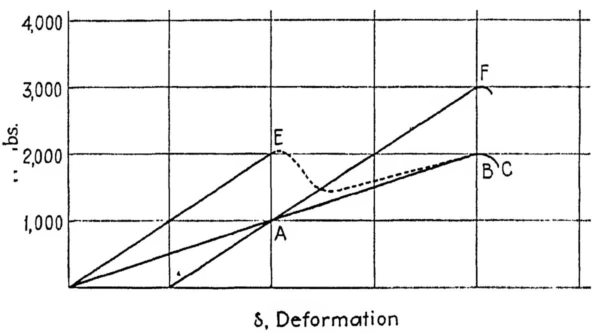


FIG. 44.—Showing effect of initial stresses.

will be $O'EBC$. Buckling in $A'B$ occurs at E when the load on this member becomes $1{,}000\sqrt{2}$ lb.; this occurs when W is 2,000 lb. If an initial stress of $500\sqrt{2}$ lb. is placed in each of the diagonal members, when the stress in member $A'B$ is reduced to zero, that in the member AB is $1{,}000\sqrt{2}$ lb. tension, the load W in this case being 1,000 lb. If W is increased to 3,000 lb., an addition of 2,000 lb., the load in AB becomes $2{,}000\sqrt{2}$ lb. and the load in $A'B$ becomes $1{,}000\sqrt{2}$ lb., the maximum in both cases. This condition is represented by the W-δ curve OAF. *The initial stress has therefore increased the strength of the structure* 50 *per cent.*

This example is an elementary case, which may not be found in an airplane structure, yet it illustrates a principle which is fundamental in design of composite structures such as that of an airplane. *Care should be exercised in this respect that the deflection*

of the major members of a structure do not cause undue stresses in minor members, thereby causing premature failures in the structure.

124. Elastic Deformation.—We have noted that the method of analysis of a statically indeterminate structure involves the physical characteristics of the members of the structure. These physical characteristics pertain particularly to the *elasticity of the material*, the *cross-sectional area*, and the *length of the members.* We should note here in passing, that there are four main elastic constants of a material: namely, the *modulus of elasticity*, *modulus of elasticity in shear*, the *bulk modulus*, and *Poisson's ratio.* The analysis of a very general type of structure would involve at least two or three of these elastic constants. However, if a structure consists of small rods, only the modulus of elasticity is involved. We shall consider first this elementary type of structure.

The definition of the modulus of elasticity, Young's Modulus, is

$$E = \frac{\text{unit stress}}{\text{unit strain}} = \frac{P/A}{y/L} = \frac{PL}{Ay} \tag{138}$$

Young's modulus of elasticity is the ratio of unit stress to unit strain. *Unit stress is the total reaction in a member divided by the area of cross section of the member. Unit strain is the total strain of the member divided by its length.* We use P here to indicate the total load in a member, A to indicate the cross section, y to indicate deformation of the member, and L to indicate the length of the member. The units are pounds and inches. We note that

$$y = \frac{PL}{AE} \tag{139}$$

Some methods of analyses of indeterminate structures involve the energy in the members of the structure. The energy stored in any member subjected to tension or compression is the average force times the deformation. The average force in this case is expressed by

$$\frac{P}{2} = F \tag{140}$$

The deformation is y. (141)

We have, therefore, energy stored,

$$u = \frac{Py}{2} \tag{142}$$

Substituting the value of y from (139) in (142), we have

$$u = \frac{P^2L}{2AE} \tag{143}$$

We note that the total energy stored in any structure built of slender struts is the summation of terms similar to equation (143).

125. Methods of Analysis.—The determination of stresses in statically indeterminate structures belongs to the general subject of *mathematical theory of elasticity.* Structures of slender rods and tubes constitute a special study of the general subject. While a great many methods of analyses have been devised, the basis is the *comparison of deformations* of the members of the structure. The *energy* method, while, as noted in equation (142), based on the deformation of members, has proved useful in simplifying the problem. The so-called *theory of least work,* or as it is sometimes called, *theory of minimum energy,* is probably the most generally used and has the most widespread application to all types of structures. Since it is not practical to present several methods in the limited space available, we shall limit the presentation to the fundamental method of comparison of deformations and to the theory of least work.

126. Method of Comparison of Deformations.—For the purpose of an elementary example, let us consider a weight W suspended from a rod of length L. Suppose that the area of the cross section of the rod is 4 sq. in. and the weight W is 4,000 lb. It is apparent in this case that, regardless of the material of the rod, the stress will be 1,000 lb. per square inch. Now let us assume that we can split the rod into two parts so that we have two rods of equal area of cross section. It is apparent, in this case, that, if the rods are of the same material, each rod will carry one-half the load, or 2,000 lb. Now continuing the illustration, suppose that we divide these rods so that one rod has 3 sq. in. and the other 1 sq. in. The first will then carry 3,000 lb. and the second will carry 1,000 lb. In other words, *the load that each rod will carry is proportional to the area of the cross section.*

Now in the case of the two rods of equal cross section, suppose that one is made of steel and the other is made of duralumin. For the purpose of the illustration, let us assume that the modulus of elasticity of the steel is 30,000,000 lb. per square inch. The modulus of elasticity of duralumin is 10,000,000 lb. per square inch. Now we note that the duralumin is three times as flexible

as the steel. This means that for any given deformation the duralumin will carry just one-third as much as the steel. The condition of the structure is such that the deformation in both metals is the same. We compare the deformations and find them equal. This means, then, that the duralumin will carry one-third as much as the steel will carry. They both together carry 4,000 lb. It is quite apparent that the steel rod will carry 3,000 lb. and the duralumin rod of the same area will carry 1,000 lb. *In general, the load carried is proportional to the modulus of elasticity.* The solution was by means of the comparison of the deformations.

Let us take note of a more general case. Suppose we assume a steel rod and a duralumin rod supporting a weight W. The steel rod has a length L_s, the duralumin rod a length of L_d. The area of cross section and the modulus of elasticity of each we shall indicate by the subscripts s and d. In the solution of this problem we note that the summation of the forces in the y-direction gives us

$$W = P_s + P_d \tag{144}$$

W equals the load carried by the steel rod plus the load carried by the duralumin rod. This equation represents the *limit of the method of statics for this problem.* It is quite obvious that we cannot solve the problem from this equation of statics alone. Since there are two unknown quantities, at least another equation is required. We obtain this by the *comparison of the deformation* of the two rods. Let this deformation be y under the applied load W. We find then, for the steel rod, that

$$y = \frac{P_s L_s}{A_s E_s} \tag{145}$$

and, for the duralumin rod, that

$$y = \frac{P_d L_d}{A_d E_d} \tag{146}$$

Thus

$$\frac{P_s L_s}{A_s E_s} \quad \frac{P_d L_d}{A_d E_d} \tag{147}$$

We note that we have two equations, (144) and (147), and two unknown terms, P_s and P_d, which enable us to solve for the unknown quantities. Solving, we have

$$P_s = \left[\frac{A_s E_s L_d}{A_s E_s L_d + L_s A_d E_d}\right] W \qquad (148)$$

and from the symmetry of the equation we have

$$P_d = \left[\frac{A_d E_d L_s}{A_d E_d L_s + L_d A_s E_s}\right] W \qquad (149)$$

In words, P_s and P_d are proportionate parts of W, depending upon the physical characteristics of the rods.

127. Example of Comparison of Deformations.—Let us consider another example. As illustrated in Fig. 45, we have a

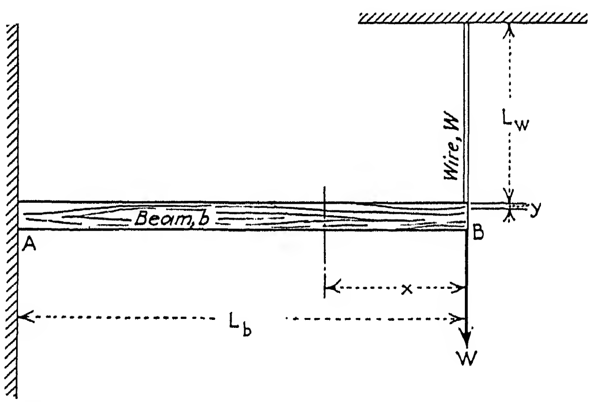


FIG. 45.—Combination of beam and wire; statically indeterminate structure.

weight W being supported by a combination of cantilever beam AB and a wire GB. From statics we have the equation,

$$W = P_b + P_w \qquad (150)$$

in which P_b is the portion of the load carried by the beam and P_w the portion carried by the wire. We designate the deflection in the direction of the load W as y. We have then the deflection y in terms of the load in the wire,

$$y = \frac{P_w L_w}{A_w E_w} \qquad (151)$$

and in terms of the deflection of the beam,

$$y = \frac{P_b L_b^3}{3E_b I_b} \tag{152}$$

in which I indicates the moment of inertia of the cross-sectional area of the beam about its neutral axis. Equating (151) and (152), we have

$$\frac{P_w L_w}{A_w E_w} = \frac{P_b L^3}{3E_b I_b} \tag{153}$$

We now have two equations, (150) and (153), and two unknown terms, P_w and P_b, from which we may find the unknown quanti-

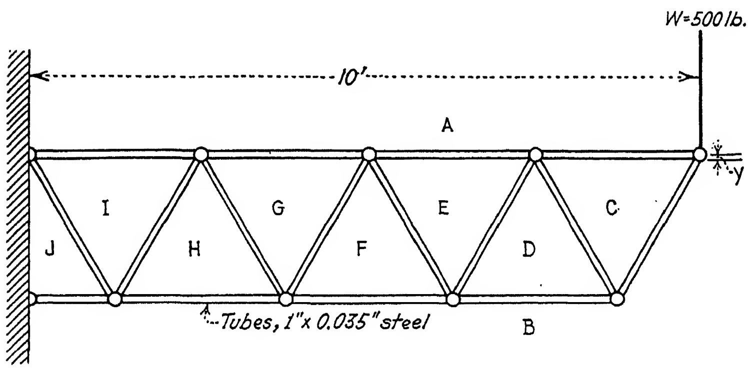


FIG. 46.—Deflection of truss by theory of least work.

ties in terms of the physical characteristics of the members and the weight W.

It should be noted here that in each one of these examples the direction of the deflection is obvious. Now in general types of structures the direction of the deflection of any particular joint may not be known. Thus *the method of deformations is applicable to cases in which the direction of the deflection is known.*

128. First Theorem of Least Work.—Consider the energy stored in a structure due to some applied load (for example, see Fig. 46).

Let us plot the curve of y as a function of W (see Fig. 47). Let us now consider the energy stored in the structure due to the added weight P and the resulting deformation y. We have

$$u = \int y dP \tag{154}$$

from which

$$\frac{du}{dP} = y \tag{155}$$

Expressed in words, *the first derivative of the energy stored in the structure with respect to the applied load is equal to the deformation of the point of application of the load in the direction of the applied load.* This is Castigliano's first theory of least work. We note in this case that the energy is the summation of the deposits of energy stored in each member of the structure. As a general rule we are not concerned with the deflection of the member

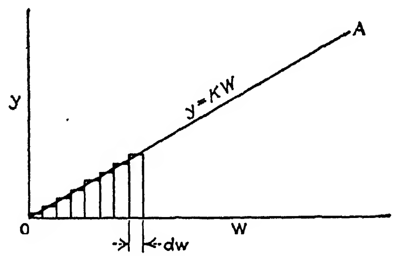


FIG. 47.—Energy stored in an elastic structure.

except as that deflection will enable us to determine the stress. It would not be difficult to find the deflection in a structure under any applied load by the application of this theorem. For example, in a structure of rods,

$$u = \sum \frac{P^2L}{2AE} \tag{156}$$

in which P is proportional to the load W causing the deflection. Thus

$$P = KW \tag{157}$$

The deflection of the point of application of W is

$$y = \frac{du}{dW} = \frac{d}{dW}\sum \frac{K^2W^2L}{2AE} \tag{158}$$

or

$$\sum \frac{K^2WL}{AE} \tag{159}$$

We note also that the internal energy is equal to the work of deformation, hence

$$\frac{P^2L}{2AE} = \frac{1}{2}Wy \tag{160}$$

from which y may be found. The substitution of KW for P in equation (160) and the proper cancellation produces an equation identical with (159). The student may find y in Fig. 46 as an exercise.

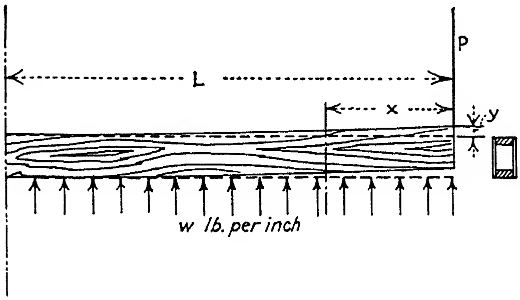


FIG. 48.

129. Torsional Deformation.—In the consideration of a structure consisting of rods, such as an airplane fuselage or engine moment, subjected to a torsional moment T, the angles of twist θ, in radians, may be computed from the equation of internal and external work of deformation by

$$\sum\frac{P^2L}{2AE} = \frac{1}{2}T\theta \tag{161}$$

in which the units used are inches, pounds, and radians.

130. Deformation under Several Loads.—If a structure is subjected to more than one load, then the deformation at the point of action of the load P in the direction of the line of action of P is the *partial* derivative of the internal energy with respect to the load P, that is,

$$\frac{\partial u}{\partial P} = y \tag{162}$$

For example, in Fig. 48, the deflection y may be computed as follows:

$$u = \int_0^L \frac{M^2}{2EI}dx \tag{163}$$

But

$$M = \frac{wx^2}{2} - Px \tag{164}$$

so that

$$u = \int_0^L \frac{\left(\frac{wx^2}{2} - Px\right)^2}{2EI} dx \tag{165}$$

Thus

$$\frac{\partial u}{\partial P} \quad \frac{2}{2EI}\left[+\frac{wx^4}{8} - \frac{Px^3}{3}\right]_0^L = y \tag{166}$$

from which

$$\frac{wL^4}{8EI} \quad \frac{PL^3}{3EI} \tag{167}$$

131. Second Theorem of Least Work.—Castigliano's second theorem is generally referred to as the "theory of least work." This theorem is a simplication and a standardization of the "method of comparison of deformations" in the determination of stresses in a statically indeterminate structure. For example, with reference to Fig. 45 and Par. 130,

$$y = \frac{\partial u}{\partial W} \quad \frac{\partial u}{\partial P_w} \quad \frac{\partial u}{\partial P_b} \tag{168}$$

$$u = \int_0^L \frac{P_b^2 x^2}{2E_b I_b} dx + \frac{P_w^2 L_w}{2A_w E_w} \tag{169}$$

Since by statics, $P_b = W - P_w$ (170)

$$\frac{\partial u}{\partial W} - \frac{WL_b^3}{3E_b I_b} \quad \frac{P_w L_b^3}{3E_b I_b} - \frac{P_b L_b^3}{3E_b I_b} \tag{171}$$

Or since by statics, $P_w = W - P_b$ (172)

$$\frac{\partial u}{\partial W} = \frac{(W - P_b)L_w}{A_w E_w} = \frac{P_w L_w}{A_w E_w} \tag{173}$$

From equations (171) and (173), we get

$$\frac{P_b L_b^3}{3E_b I_b} = \frac{P_w L_w}{A_w E_w} \tag{174}$$

which with equation (170) permits of a solution as noted in equations (150) and (153).

Now this process may be much simplified if we consider the wire removed and P_w as an outside force. In this case

$$ \tag{175}$$

Noting that

$$P_b = W - P_w \tag{176}$$

we obtain

$$\frac{\partial u}{\partial P_w} \qquad -\frac{(W - P_w)L_b}{3E_bI_b} \qquad -\frac{P_bL_b}{3E_bI_b} \tag{177}$$

And since it is obvious that the deformation of the wire in the direction of P_w is P_wL_w/A_wE_w, equation (174) may be written.

A further simplification may be made by the following process: Note that

$$\frac{\partial u}{\partial P_w} \qquad -\frac{P_wL_w}{A_wE_w} \tag{178}$$

from which

$$\frac{\partial}{\partial P_w}\left(u + \frac{P_w^2L_w}{2A_wE_w}\right) = 0 \tag{179}$$

Since u is the energy in the beam and $P_w^2L_w/2A_wE_w$ is the energy in the wire, the sum of the two is the total energy in the structure, which we may designate as U in the equation

$$\frac{\partial U}{\partial P_w} = 0 \tag{180}$$

This gives us an equation identical with equation (174) which, with the equations of statics, is sufficient to solve for the unknown values. *Equation* (180) *is the basic equation of the "theory of least work."*

Equation (179) is applicable to a general type of structure; for example, u is the energy in all members of the structure except the member w under consideration. Now the deformation of the structure due to P_w, considered as an outside force, is

$$\frac{\partial u}{\partial P_w} = -y_w = -\frac{P_wL_w}{A_wE_w} \tag{181}$$

or, as noted in equation (179), $\partial U/\partial P_w = 0$.

132. Steps in the Application of the Theory of Least Work as Applied to Structures of Pin-ended Rods.—*The first step* is the examination of the structure and determination of how many members must be removed in order that the structure may be left statically determinate. Also, it should be determined which members should be considered redundant in order to simplify the methods of statics as much as possible.

The second step is to compute the loads in the members of the statically determinate structure, preferably by the analytical method of statics. It is assumed that the redundant members have been removed. The graphical method in this connection is not so desirable because of the unknown loads in the removed redundant members, for example, P_1, P_2, etc. In determining the loads in the statically determinate structural members, the loads in the redundant members P_1, P_2, etc., are considered as outside applied loads. It therefore is obvious that the load in any member will be expressed in terms of the outside loads W_1, W_2, etc., and the loads P_1, P_2, etc., in the redundant members.

The third step is to write the equation of energy for all the members of the structure including the redundant members. Since the loads in the members are expressed in terms of the loads in the redundant members, the energy equation then will involve only the unknown loads P_1, P_2, etc., of the redundant members.

The fourth step is to take the partial derivative of this energy equation with respect to each of the forces P_1, P_2, etc., in the redundant members and to equate each of these resulting equations to zero. This will produce as many additional equations as there are unknown loads P_1, P_2, etc.

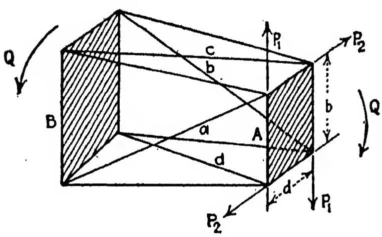


FIG. 49.—Torsion in a fuselage bay.

The fifth step is to solve these additional equations simultaneously for the unknown values P_1, P_2, etc.

See Sec. IV, Chap. 18, for practical applications.

133. Example of Second Theory of Least Work.—Consider an example of torsion in a fuselage structure of the conventional tube-and-wire type. For simplicity we assume one bay only. In Fig. 49 we have one bay of the fuselage. We assume the torque Q applied to bulkhead B. We assume that these bulk-

heads are absolutely rigid. (You will note that the problem will be considerably more complicated if we assume the bulkhead elastic.)

Let us assume that the wires which do not carry a part of the load are removed. Therefore, we have wires a, b, c, and d as resisting the applied torque. Now we note that the wires a and c may be removed without impairing the stability of the structure. However, upon removal of these wires, of course, a side load cannot be resisted, that is, a side load which normally would be taken by these two wires. We also note that either pair of wires may be removed and still the structure is rigid. However, if the wire a is removed, c carries no load, since the summation of the forces in the x- and y-directions must be zero.

Let us divide the torque Q into couples such as P_1d and P_2b. We note that

$$Q = P_1 d + P_2 b \tag{182}$$

Our equation of energy is

$$U = \frac{P_a^2 L_a}{2A_a E_a} + \frac{P_b^2 L_b}{2A_b E_b} + \frac{P_c^2 L_c}{2A_c E_c} + \frac{P_d^2 L_d}{2A_d E_d} \tag{183}$$

If the fuselage is assumed to be symmetrical with respect to the axis, we have

$$P_b = P_d, \qquad L_b = L_d, \qquad A_b = A_d$$

and if we assume

$$E_a = E_b = E_c = E_d = E \tag{184}$$

our energy equation reduces to the following form:

$$U = \frac{P_a^2 L_a}{A_a E} + \frac{P_b^2 L_b}{A_b E} \tag{185}$$

If we let K_1, K_2, etc., be constants which may be determined analytically or graphically, we have

$$P_a = K_1 P_1, \qquad \text{and} \qquad P_b = K_2 P_2 \tag{186}$$

If we substitute these values of P_a and P_b in equation (185), we have

$$U = \frac{K_1^2 P_1^2 L_a}{A_a E} + \frac{K_2^2 P_2^2 L_b}{A_b E} \tag{187}$$

Now substituting the value of P_1 from equation (182) in equation (187), we have

$$U = \frac{K_1^2\left(\frac{Q - P_2 b}{d}\right)^2 L_a}{A_a E} + \frac{K_2^2 P_2^2 L_b}{A_b E} \tag{188}$$

Taking the partial derivative of U with respect to P_2 and equating to zero, we have

$$\frac{\partial U}{\partial P_2} = -\frac{2bK_1^2(Q - P_2 b)L_a}{dA_a E} + \frac{2K_2^2 P_2 L_b}{A_b E} = 0 \tag{189}$$

from which equation we can find P_2.

Now we note that if we consider a number of the bays in the fuselage, together with the cross wires in the bulkhead, we find that there is a redistribution of stress at each bay so that the load carried by each wire and produced by the applied torque Q is a function of the physical characteristics of every member in the fuselage structure.

134. General Equation for Statically Indeterminate Structure Composed of Pin-ended Rods.—The work involved in the application of the various methods to the evaluation of forces in redundant members of a statically indeterminate structure, where the degree is greater than the first or second, involves considerable detailed mathematics. Now, if we develop a general method whereby the results may be tabulated in a convenient form, considerable work and energy may be saved. For this purpose let us consider the structure in Fig. 50. This structure has three redundant members which we may consider to be 1, 2, and 3. Now for all practical purposes we may assume this to be a general type of structure. To simplify, in general, the designation of members, let us designate the assumed redundant member by numerals and the assumed statically determinate members by letters of the alphabet. Let us remove the redundant members and insert in the place of each of them a force of 1 lb. Now, let us proceed to apply the methods of statics to determine the loads in the statically determinate members a, b, and c as follows:

For example, taking the member a, we write the following equation:

$$P_a = P_{0a} + P_1 K_{1a} + P_2 K_{2a} + P_{3a} K_{3a} + \cdots \tag{190}$$

in which P_a is the total load in the member a; P_{0a} is the load in the member a in the statically determinate structure induced by the weights w_1, w_2, w_3, and w_4; K_{1a} is the load induced in the member a of the statically determinate structure by a 1-lb. force substituted for the member 1; P_1 is the total load in member 1, which is to be determined; K_{2a} is the load in the member a due

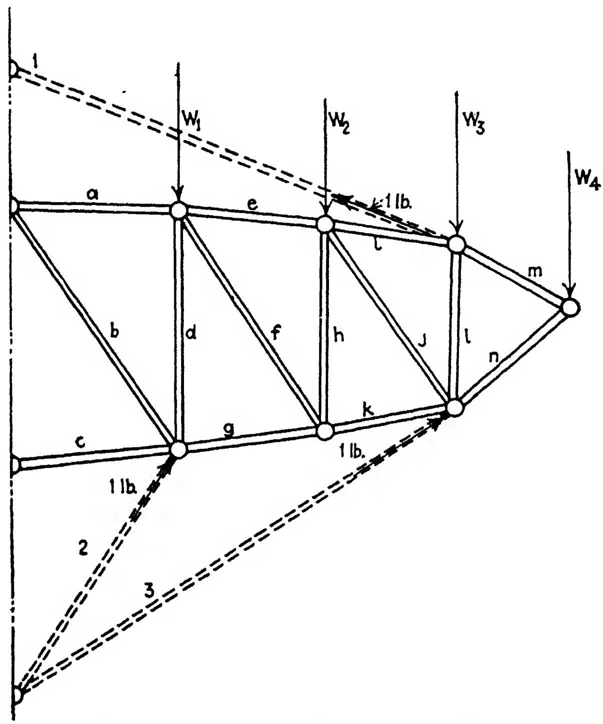


FIG. 50.—Statically indeterminate structure.

to the 1-lb. load substituted for member 2; and P_2 is the total load in the member 2. Thus, we may add terms for each redundant member. It will be noted that constants K_{1a}, K_{2a}, and K_{3a} are coefficients which may be determined by methods of statics.

We write a similar equation for the statically determinate member b:

$$P_b = P_{0b} + P_1K_{1b} + P_2K_{2b} + P_{3b}K_{3b} + \cdot \cdot \cdot \cdot \qquad (191)$$

We note in this equation that we have the same terms as in equation (190), but that these terms refer to member b rather

than to member a. Likewise, we have the same condition for member c:

$$P_c = P_{0c} + P_1K_{1c} + P_2K_{2c} + P_3K_{3c} + \cdots \qquad (192)$$

As in general we have many possible equations of statics, those three equations will not be the limit of the alphabetically subscripted terms. It will also be noted that we may have any number of redundant members. Now let us write the equation of energy as follows (in this equation let us write the energy in the redundant members first):

$$U = \frac{P_1^2L_1}{2A_1E_1} + \frac{P_2^2L_2}{2A_2E_2} + \frac{P_3^2L_3}{2A_3E_3} + \qquad (193)$$

$$+ \frac{P_a^2L_a}{2A_aE_a} + \frac{P_b^2L_b}{2A_bE_b} + \frac{P_c^2L_c}{2A_cE_c} + \cdots \qquad (194)$$

Now, substituting in this equation the values of P_a, P_b, and P_c, and P_1K_1, P_2K_2, and P_3K_3 for P_1, P_2, and P_3, respectively, ($K_1 = K_2 = K_3 = 1$), we have

$$U = \frac{(P_1K_1)^2L_1}{2A_1E_1} + \frac{(P_2K_2)^2L_2}{2A_2E_2} + \frac{(P_3K_3)^2L_3}{2A_cE_c} + \qquad (195)$$

$$+ \frac{(P_{0a} + P_1K_{1a} + P_2K_{2a} + P_3K_{3a} + \cdots)^2L_a}{2A_aE_a}$$

$$+ \frac{(P_{0b} + P_1K_{1b} + P_2K_{2b} + P_3K_{3b} + \cdots)^2L_b}{2A_bE_b} \qquad (196)$$

$$+ \text{(Similar terms for } c, d, e, \text{ etc.)}$$

Applying the theory of least work, we take the partial derivative of U with respect to each redundant member and equate each of these to zero. In these equations let us designate the physical constants of the members by an appropriate letter; for example, let

$$\frac{L}{AE} = B, \qquad \frac{L_1}{A_1E_1} = B_1 \qquad (197)$$

Taking the partial derivative, we have

$$\frac{\partial U}{\partial P_1} = P_1K_1^2B_1 + K_{1a}(P_{0a} + P_1K_{1a} + P_2K_{2a} + P_3K_{3a} + \cdots)B_a$$
$$+ K_{1b}(P_{0b} + P_1K_{1b} + P_2K_{2b} + P_3K_{3b} + \cdots)B_b +$$
$$+ K_{1c}(P_{0c} + P_1K_{1c} + P_2K_{2c} + P_3K_{3c} + \cdots)B_c$$
$$+ \cdots = 0 \qquad (198)$$

Subdividing this equation, we have

$$\begin{aligned}\frac{\partial U}{\partial P_1} = {} & P_1K_1^2B_1 + (P_{0a}K_{1a}B_a + P_{0b}K_{0b}B_b + P_{0c}K_{0c}B_c) \\ & + (P_1K_{1a}^2B_a + P_1K_{1b}^2B_b + P_1K_{1c}^2B_c) + (P_2K_{1a}K_{2a}B_a \\ & + P_2K_{1b}K_{2b}B_b + P_2K_{1c}K_{2c}B_c) \\ & + (P_3K_{1a}K_{3a}B_a + P_3K_{1b}K_{3b}B_b + P_3K_{1c}K_{3c}B_c) \\ & + \cdots = 0 \qquad (199)\end{aligned}$$

This may be written

$$\frac{\partial U}{\partial P_1} = P_1K_1^2B_1 + \sum_a^n P_0K_1B + P_1\sum_a^n K_1^2B + P_2\sum_a^n K_1K_2B + P_3\sum_a^n K_1K_3B = 0 \qquad (200)$$

Writing a similar equation for the partial derivative with respect to P_2, we have

$$\frac{\partial U}{\partial P_2} = P_2K_2^2B_2 + \sum^n P_0K_2B + P_2\sum^n K_2^2B + P_1\sum^n K_1K_2B + P_3\sum_a K_2K_3B = 0 \qquad (201)$$

and for P_3 we have

$$\frac{\partial U}{\partial P_3} = P_3K_3^2B_3 + \sum_a^n P_0K_3B + P_3\sum_a^n K_3^2B + P_2\sum_a^n K_3K_2B + P_1\sum_a^n K_1K_3B + \cdots = 0 \qquad (202)$$

We can now write these equations for any number of redundant members as 1, 2, 3, 4, 5, etc. These equations (200), (201), and (202) are the equations to be solved for the unknown loads P_1, P_2, and P_2 in the redundant members. We note that there is one equation for each redundant member, enabling the solution to be obtained. Now let us note how we would tabulate

the quantities for these equations. For example, consider Table XVI.

TABLE XVI.—TABULATION OF CALCULATIONS

Member	B	K_1	K_2	K_3	P_0	P_0K_1B	P_0K_2B	P_0K_3B	K_1^2B	K_2^2B	K_3^2B	K_1K_2B	K_1K_3B	K_2K_3B

135. The Application of the General Method.—*The first step* is to assume the redundant members in the structure, as, for example, members 1, 2, etc. These redundant members are removed and 1-lb. loads are inserted as substitutions for the end reactions of the members.

The second step is to determine by the methods of statics the loads due to the outside applied loads in the members of this structure (with the redundant members removed). These loads are designated as P_0; for example, if the statically determinate members are a, b, c, these loads will be P_{0a} P_{0b}, P_{0c}.

The third step is to determine, by methods of statics, the loads in the members of the statically determinate structure which are due to the 1-lb. load substituted for the member 1. These loads will be, for example, K_{1a}, K_{1b}, and K_{1c}.

The fourth step is to determine the loads in the members of the statically determinate structure which are due to the 1-lb. load substituted for the member 2, as, for example, K_{2a}, K_{2b}, K_{2c}, etc.

The fifth step is to compute the constant B for each of the members.

The sixth step is to complete Table XVI.

The seventh step is to write the equations as indicated by equations (200), (201), and (202), one equation for each unknown quantity.

The eighth step is to solve the resulting equations simultaneously.

136. Stresses above Proportional Limit.—In a statically indeterminate structure, for example, as shown in Fig. 51*a*, in which stresses in the members exceed the proportional limit, there is a tendency for the *unit stresses in all members to approach the same value*, provided the members are of the same material.

This can be shown by the theory of least work through successively increasing the load W by increments of 100 lb., using the modulus of elasticity for each increment as dictated by the total stress in the member as shown by f in Fig. 51b.

This may also be visualized. For example, suppose the member B becomes stressed beyond the elastic limit before A and C. Its effective modulus of elasticity becomes less, which means that the member stretches more appreciably with less increments

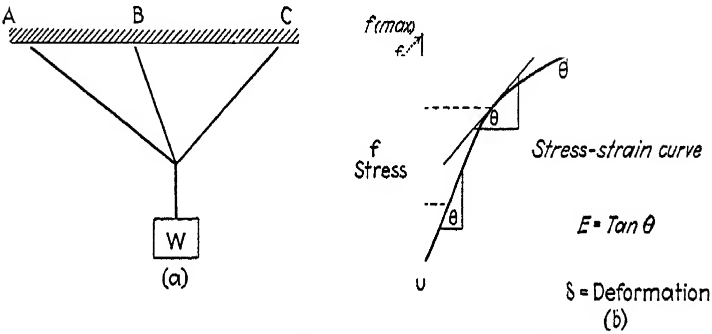


FIG. 51.—Stress above proportional limit.

of load. The stress in the member tends to become constant. In the meantime, however, as the load W is increased, the stresses in A and C are increasing. If one member does not break, it is obvious that they will each reach $f_{(max.)}$ as shown on the curve.

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CHAPTER X

BEAMS AND STRUTS

137. Modulus of Rupture.—The theoretical breaking strength of a beam subjected to bending is given by the formula:

$$f_b = \frac{Mc}{I} \tag{203}$$

in which f_b is the stress in the outer fiber; M is the applied bending moment, c is the distance from the neutral axis of the cross-sectional area to the outermost fiber, and I is the moment of inertia of the cross-sectional area with reference to the neutral axis. The student is advised to study the derivation of this formula with special reference to the *assumptions* on which the derivation is based. If the formula were valid beyond the elastic limit, the value of f_b for rupture would agree reasonably well with the ultimate strength of the material. However, since the formula is thus used incorrectly, the value of f_b is found to agree with neither the ultimate tensile strength nor the compressive strength of the material. *This value of f_b cannot be regarded as a physical constant, but as a figurative value, valuable mainly for comparative purposes.* This figurative value is known as the *modulus of rupture* of the material.

138. Application of the Modulus-of-rupture Formula.—It may be noted that the modulus-of-rupture formula is useful only in applying results from *model tests* to the design of beams. This means that, for an accurate application of the formula, it is necessary to have the results of experiments on beams similar to the beams which are being used in airplane design. The fiber stress f_b which we are capable of developing in a beam will depend upon the configuration of the cross section of the beam. For example, let us consider Fig. 52, in which we have represented an I-beam and a box beam.

If these beams are subjected to a bending load, the question of *instability* enters, because of the thin webs and flanges. The fibers in the beam may not fail but they may buckle, developing

thereby much less strength than they would develop if they were properly supported. It is therefore apparent that the fiber stress f_b allowed in beams of this type would be less than the fiber stress allowed in a solid beam.

In many types of beams in thin sheet-metal construction, the modulus of rupture is displaced by the compressive stress which causes buckling of the thin sheet in the outermost fibers of the compression flange of the beam. If we designate this buckling stress by

$$\frac{P}{A} = \frac{\text{axial load}}{\text{area of cross section under compression}}$$

we have

$$\frac{P}{A} = \frac{Mc}{I} \tag{204}$$

139. Calculation of Shear in the Web of a Beam.—In the consideration of the design of a wing beam, special attention

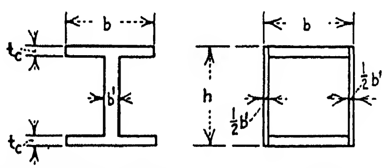


FIG. 52.—Standard dimensions of I-beam and box beam.

must be given to the design of the thin web as in a box beam or in an I-beam. The load which occurs in the web of a wing beam is mostly shear. As a matter of fact, the webs of a wing beam are generally neglected in the calculation of tension and compression. They are thus designed exclusively for the resistance of the shear in the beam.

Consider the beam in Fig. 53, subjected to the bending moment

$$M = Wx \tag{205}$$

We have, as shown in Fig. 54, then, the total shearing load on the beam at A as F_1, in which

$$F_1 = \frac{Wx_1}{d} \tag{206}$$

Now at a section B we note that

$$Wx_2 \tag{207}$$

and that the resultant shearing stress on the increment Δx is

$$F_s = F_1 - F_2 \tag{208}$$

or

$$F_s = \frac{M_A - M_B}{d} \tag{209}$$

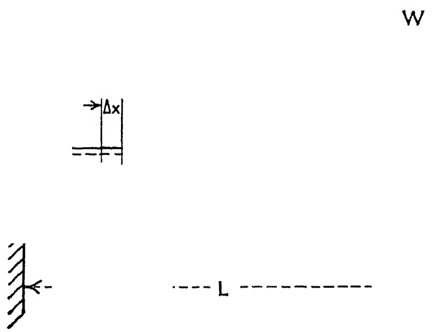


Fig. 53.

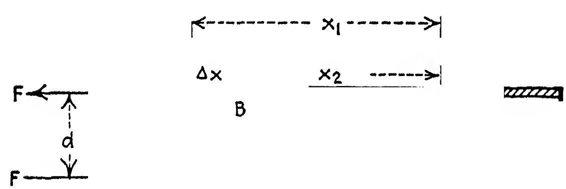


Fig. 54.

We note that the unit shear which we designate as s is equal to the total shear F_s divided by the area,

$$\frac{F_s}{b\Delta x} \tag{210}$$

Here b is the total width of the web material. Now, substituting the value of F_s from equation (209) into equation (210), we have

$$s = \frac{M_A - M_B}{bd\Delta x} = \frac{\Delta M}{bd\Delta x} \tag{211}$$

or, in the limiting form, we find s for this particular case,

$$s = \frac{dM}{dx(db)} = \frac{V}{bd} \tag{212}$$

Now let us consider the general case for the solid beam. In Fig. 55 consider the section of a beam which is bent. At section A the moment is M_A; at section B the moment is M_B. Now at A the total shearing stress at any point is

$$F_A = \sum^{c}_{v} P_{ta} \tag{213}$$

in which P_t is the tensile or compressive load on an area dA of the cross section. Likewise

$$F_b = \sum^{c}_{v} P_{tb} \tag{214}$$

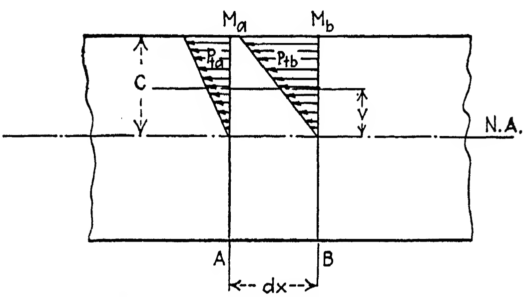


FIG. 55.—Shear in a bent beam.

The resultant shearing stress on the section of length dx is

$$F_A - F_B = \sum^{c}_{v} (P_{ta} - P_{tb}) \tag{215}$$

The unit shear in pounds per square inch of shearing surface is

$$s = \frac{F_A - F_B}{bdx} = \frac{\sum^{c}_{v} P_{ta} - P_{tb}}{bdx} \tag{216}$$

Note that

$$P_{ta} = f_a dA \tag{217}$$

and

$$P_{tb} = f_b dA \tag{218}$$

Now from the modulus-of-rupture formula, equation (203) we have

$$f = \frac{Mv}{I} \tag{219}$$

so that

$$f_a = \frac{M_a v}{I} \tag{220}$$

and

$$f_b = \frac{M_b v}{I} \tag{221}$$

in which we assume the beam to be uniform in cross section.

Now, substituting the values of equations (220) and (221) in equations (217) and (218), and in turn substituting these values in equation (216), we have

$$s = \sum_{v}^{c} \frac{M_A v dA - M_B v dA}{I b dx} \tag{222}$$

And, since the difference between the two moments is the differential moment, we have expressed the equation in an integral form,

$$s = \frac{1}{Ib} \sum_{v}^{c} \frac{(M_A - M_B) v dA}{dx} = \frac{1}{Ib} \int_v^c \frac{dM}{dx} v dA \tag{223}$$

Now, dM/dx is the shear on the beam at the point x, and this term is independent of the variable v; therefore, this equation may be written

$$= \frac{V}{Ib} \int_v vdA \tag{224}$$

in which V is the total shear on the beam at the point x. The integral in equation (224) is the statical moment of an area about a line through the point at which we desire to obtain the shear; in general this is the neutral axis of the beam, so that

$$\int_v^c vdA = Q \tag{225}$$

Thus

$$\frac{VQ}{Ib} \tag{226}$$

140. Allowable Shear in Web of a Beam.—While equation (226) enables us to calculate the stress, it does not enable us to determine whether or not the webs will carry the stress. Thus before the beam can be designed, it will be necessary for us to know what stress s may be allowed in the webs. Obviously s will depend upon the buckling condition of the web; that is, if the web is very thin and not very well braced, it is quite obvious

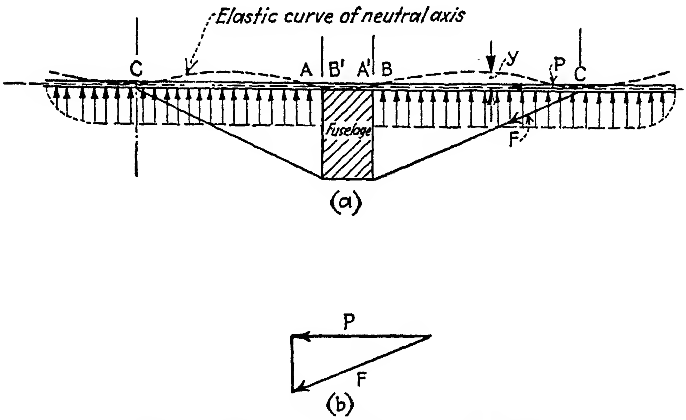


FIG. 56.—Secondary stresses in a wing beam.

that it will buckle easily, and very little shearing stress can be resisted. However, if the web is fairly thick and well braced, the shearing stress s may be fairly high. In certain cases of metal beams where thin webs are used, the shearing stress in the webs is resisted entirely by a diagonal tensile stress, in which case it is necessary to brace the flanges of the beam by bulkheads to prevent the beam from collapsing.

The tensile lines in the thin web will assume approximately a 45-degree direction. Now it is quite obvious in this case that s is not calculated as a shear stress but as a tensile stress in the thin webs and as a shear stress in the rivets which attach the thin webs to the flanges (see Par. 229 for further consideration of this problem).

141. Secondary Stress Due to Flexure.—The flexibility of an airplane structure alters very greatly the stress in the members of

the structure, and therefore it is of prime importance to be able to calculate the effect of the flexure in the members. *The stress due to a flexure of the structure is called a secondary stress.* In many instances the secondary stress in a wing beam and in other types of beams in an airplane structure is greater than the primary stress in the beam. This is particularly true when the beam acts not only as a beam but also as a column, as, for example, the wing beam in a biplane or in a braced monoplane. When the beam is bent by a side load, and then an axial load is applied to the beam, a *secondary* stress is induced by this load, for example, as noted in Fig. 56. We have in this figure the flexure of the wing beam due to the flying load indicated by y. The axial load on this beam is a component P of the load in the flying wire. The secondary moment in this case is Py. If the deflection is very great, moment Py may be greater than the bending moment due to the air-load.

142. Calculation of Flexure of a Beam.—Now let us consider the calculation of this flexure. At any point x along the beam, the beam has a certain radius of curvature. The radius of curvature ρ is given by the equation

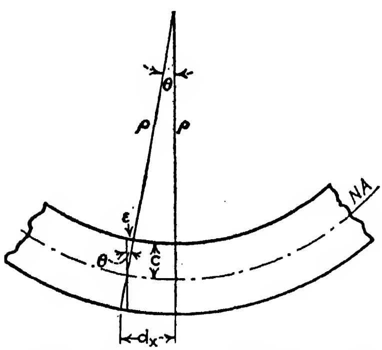


FIG. 57.—Radius of curvature of a bent beam.

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \tag{227}$$

For small curvatures the term $(dy/dx)^2$ is assumed zero, so that

$$\rho = \frac{1}{d^2y/dx^2} \tag{228}$$

This equation is invariably used for deflection calculations with accurate results. It remains therefore for us only to find the value of ρ, the radius of the curvature, in terms of known quantities such as the bending moment, modulus of elasticity, and the moment of inertia of the section. For this purpose let us consider Fig. 57, which is a section of a beam which is bent. We have, from the definition of E,

$$f = E\frac{\epsilon}{dx} \tag{229}$$

Now

$$\rho : dx = c : \epsilon \tag{230}$$

from which equation we find that

$$\frac{\epsilon}{dx} \quad \frac{c}{\rho} \tag{231}$$

Now, substituting this value in equation (229), we find that

$$f = E\frac{\epsilon}{\rho} \tag{232}$$

which, when substituted in equation (203), gives us

$$E\frac{c}{\rho} = \frac{Mc}{I} \tag{233}$$

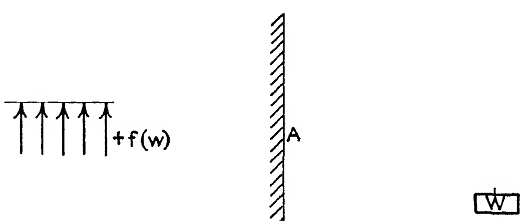
from which

$$\frac{1}{\rho} \quad \frac{M}{EI} \tag{234}$$

or

$$EI\frac{d^2y}{dx^2} = M \tag{235}$$

Now, in the application of formula (235), great care should be exercised to see that the conditions relative to the sign of these



(*a*) (*b*)

FIG. 58.—(*a*) Positive load; (*b*) positive shear.

terms are correct. In some cases the definition of our convention of signs, unless very carefully observed, will produce erroneous results. In such a case we should note that our conventions are matters of definition; and the error involved might not be an error of theory, but merely an error of definition.

143. Convention of Signs.—We assume a *positive load* with respect to the x and y coordinate plane as *upward* (Fig. 58*a*).

We define a shear in the following manner: *If the section to the right shears downward, the shear is said to be positive.* For example, in Fig. 58*b*, the load W induces positive shear in the beam AB.

A positive moment is that moment which produces compression in the upper fiber of the beam. This requires that the radius of curvature for a positive moment shall be upward, that is, in the positive direction of y. These conventions are purely matters of definition, but, after the assumptions are made for a problem, we cannot change them for that problem. We also note that

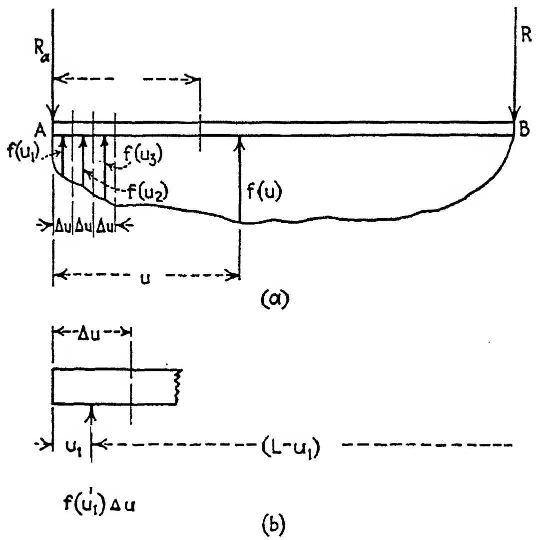


FIG. 59.—Bending moment for distributed load.

these conventions may not conform to the conventions of shear and bending moment, etc., in problems of statics; it should be understood that any connection between the two would be a matter of definition rather than a matter for logical thought.

144. Bending Moment Due to Distributed Load.—In Fig. 59 is a beam simply supported at the ends, subjected to a non-uniformly distributed load $f(u)$. We designate the section at which the moment is to be determined by x. The length of the beam is L. We determine first the reaction R_a at A. In order to determine the reaction at A we sum the moments about the point B as noted:

$$\Sigma M_B = 0 \tag{236}$$

Since the load is nonuniform, we divide it up into increments, the length of each increment being the fixed quantity Δu. We note that the load for a length Δu is the load per unit length $f(u)$ times the length Δu. That is,

$$p = f(u)\Delta u \tag{237}$$

Now let us consider the first increment, that next to station A. Let us designate the average value of the unit load for the increment by $f(u_1)$. The total load is

$$p_1 = f(u_1)\Delta u \tag{238}$$

The moment due to this particular increment is (see Fig. 59*b*):

$$\Delta M_b = p_1(L - u_1)\Delta u = f(u_1)(L - u_1)\Delta u \tag{239}$$

The moment at B, due to all the increments of loading, is

$$\begin{aligned}\Sigma M_B = f(u_1)(L - u_1)\Delta u + f(u_2)(L - u_2)\Delta u + f(u_3)\\ (L - u_3)\Delta u + \cdots - R_a L = 0\end{aligned} \tag{240}$$

Thus, we have from this equation, by approaching the limiting value du,

$$R_a = \frac{1}{L}\int_0^L f(u)(L - u)du \tag{241}$$

The bending moment at x, assuming the beam to be cut at the point x, and considering the cantilever extension to the left, is the sum of the bending moment due to the applied nonuniform load to the left of the section and to the concentrated reaction R_a. We therefore have

$$M_x = -R_a x + \int_0^x f(u)(x - u)du \tag{242}$$

in which case we use the negative sign in front of R_a because the lead R_a causes an extension in the upper fiber of that portion of the beam. Now, substituting the value of R_a as found in equation (241) in equation (242), we have

$$M_x = -\frac{x}{L}\int_0^L f(u)(L - u)du + \int_0^x f(u)(x - u)du \tag{243}$$

This equation applies to any type of loading, for example, if we assume the load to be uniform and equal to w, that is,

$$f(u) = w \tag{244}$$

we can evaluate the integrals of equation (243). Thus we find that

$$M_x = -\frac{x}{L}\int_0^L w(L - u)du + \int_0^x w(x - u)du \qquad (245)$$

Integrating this we get

$$M_x = -\frac{wL^2x}{L} + \frac{wL^2x}{2L} + wx^2 - \frac{wx^2}{2}$$
$$= -\frac{wLx}{2} + \frac{wx^2}{2} \qquad (246)$$

145. Bending Moments at Ends of Bay.—In Fig. 60 is represented a simply supported beam subjected to an applied moment

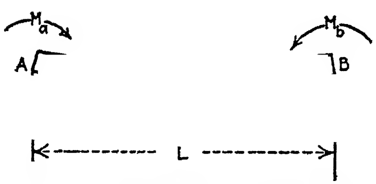


Fig. 60.—Bending moment at support.

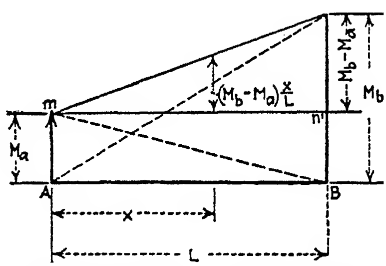


Fig. 61.—Variation of bending moment due to moments at supports.

at each end, namely M_a and M_b. Plot the vectors for these moments (see Fig. 61). These vectors, of course, would occur in a plane perpendicular to the plane of the page, but we may assume that we have rotated our axis so that the vectors appear in the same plane. If M_b is zero, the moment in the beam, that due to M_a, is directly proportional to the distance from the point B. Likewise, the effect of the moment due to M_b would be proportional to the distance from A. Now the moment at x is the sum of the moments due to M_a and M_b as represented by the line mn. The value of the moment at any point, it will be observed, is

$$M_x = M_a + \frac{(M_b - M_a)x}{L} \qquad (247)$$

146. General Moment Equation.—We find then that the total bending moment which may occur on the spars of an airplane is

$$M_x = M_a + \frac{(M_b - M_a)x}{L} - \frac{x}{L}\int_0^L f(u)(L - u)du$$
$$+ \int_0^x f(u)(x - u)du - Py \quad (248)$$

In this equation the M_a and the M_b are determined by the *three-moment equation* if the beam is continuous, or by the loading on the cantilever overhang in the portion in which the beam is discontinuous.

147. Relation Between Load, Shear, Moment, and Deflection.—The relation between load, shear, bending moment, and deflection is useful in stress analyses in connection with struts which have side load, and in connection with wing beams. We may note that the shear V is

$$V = \int wdx \quad (249)$$

The bending moment M is

$$M = \int\int wdxdx = \int Vdx \quad (250)$$

And since

$$EI\frac{d^2y}{dx^2} \quad M \quad (251)$$

we have the slope i,

$$\frac{dy}{dx} = i \quad \int\int\int\frac{wdxdxdx}{EI} = \int\frac{Mdx}{EI} \quad (252)$$

Likewise, the deflection is

$$y = \int\int\int\int\frac{wdxdxdxdx}{EI} = \int idx \quad (253)$$

148. Differentiating the Moment Equation.—In general the reverse of the procedure in equations (249) to (253) would not be possible; for one reason, the loading may be composed of concentrated loads or other discontinuous types of loading so that we encounter discontinuous functions, and for another reason, the constants of integration may be missing. For example, the operation d^2M/dx^2, when applied to

$$M_x = -\frac{4L^2}{\pi^2}W_0 \cos\frac{\pi}{2L}x + C_1x + \quad (254)$$

and to

$$M_x = -\frac{4L^2}{\pi^2}W_0 \cos\frac{\pi}{2L}x \tag{255}$$

will give the same loading curve,

$$w = w_0 \cos\frac{\pi}{2L}x \tag{256}$$

Equation (254) is the correct equation, in which C_1 and C_2 may be evaluated by the two conditions:

When $x = L$, shear $= 0$
When $x = L$, moment $= 0$

We may also note that, whereas, from a mathematical standpoint, these discontinuous functions may be apparently difficult, from a practical standpoint they offer the engineer no practical difficulty.

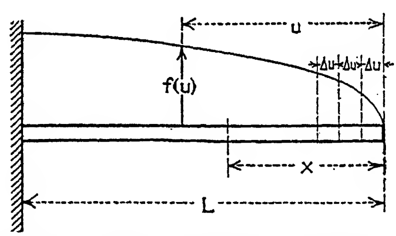


FIG. 62.—Distributed load for wing beam.

149. Integration of the Load Curve.—Note some of the applications of equations (249) to (253). Consider, for example, a cantilever beam, shown in Fig. 62, which is subjected to a known distributed load which we may designate, as a function of u, as $f(u)$. Designate a portion of the beam at which point the bending moment, slope, or deflection is to be calculated as x. We notice, in this case, that we use two variables, u and x. The two variables may at certain times indicate the same point on the beam. However, the variable x indicates the point at which we are to determine the shear, the bending moment, or other qualities, while u indicates the portion of the load which is being considered in connection with the load. For example, suppose that we are to find the shear at point x_1. x, then, is fixed as x_1, yet the shear at point x_1 will be the summation of the

loading $f(u)$ from $x = 0$ to $x = x_1$. This may be indicated as in equation (257),

$$V_x = f(u_1)\Delta u + f(u_2)\Delta u + f(u_3)\Delta u + \qquad (257)$$

in which we note that $f(u)$ is the magnitude of the load per unit of length, while Δu is the length into which the beam is divided. We note in this case that

$$\Delta u = \frac{x}{n}$$

where $n = 1, 2, 3$, etc.

Equation (257) defines a definite integral which may be written

$$V_x = \int_0^x f(u)du \qquad (258)$$

which, expressed in words, means that the shear at the point x is equal to the total load on the beam from $x = 0$ to $x = x_1$. For example, if in equation (258) $f(u) = w$, we have

$$V_x = \int_0^x wdu = wx \qquad (259)$$

We note that the bending moment is

$$M_x = [f(u_1)\Delta u](x - u_1) + [f(u_2)\Delta u](x - u_2) + [f(u_3)\Delta u](x - u_3) + \cdots \qquad (260)$$

or, expressed as an integral, is

$$M_x = \int_0^x f(u)(x - u)du$$

Now, as an example of this, let us again assume the loading uniform, that is, $f(u)$ to be replaced by w. We have, therefore,

$$M = \int_0^x w(x - u)du = \left[wxu - w\frac{u^2}{2}\right]_0^x = \frac{wx^2}{2}$$

150. Evaluating the Beam Integrals.—Graphical methods have been devised for evaluating these integrals. However, one is less inclined to make a mistake, and possibly the solution is simplified, by making use of the application of the definition of the definite integral, as, for example, in the following illustration pertaining to a cantilever beam:

In Fig. 63*a* we have plotted the load w to scale. In Fig. 63*b* we have plotted the shear to the proper scale. Now, in evaluating the integral graphically, we note that before we can start to plot the shear curve we must know the value of the shear at some point from which the curve is started. This is the graphical

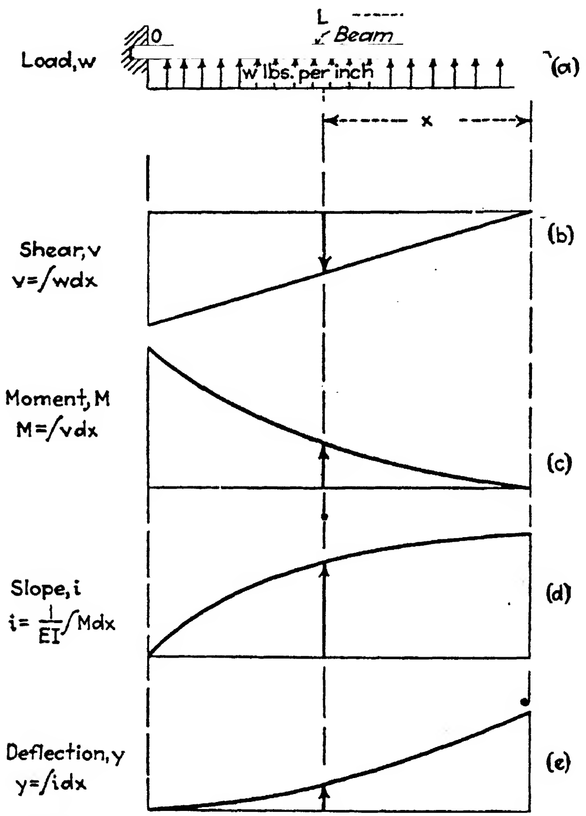


FIG. 63.—Relationship between beam load, shear, moment, slope and deflection.

equivalent of evaluating the constant of integration. For example, at point O the value of the shear is not known, but at point C it is apparent that its value is zero. We start plotting the integral at point C where the shear is known, thus obtaining the shear curve BC. We note that the shear at any point x is the summation of the loading from A to x. That is, the value of the shear ordinate at the point x is the total sum of the loading

from A to x. The negative value of the shear does not result from the integration, but instead results from the definition of a positive shear. If the cantilever beam had been turned toward the left by definition, the shear would have been positive.

In plotting the bending-moment curve, as noted in Fig. 63*c*, for the purpose of determining where the curve shall be started, we notice that the bending moment at E is known; that is, the bending moment at E is zero. We also note that, from the nature of the loading, the bending moment will be positive. We therefore have the bending-moment curve ED in c, in which the ordinate at any point x is the summation of the shear from C to x.

We note that in the slope curve the value of the slope is known at point O. We therefore start plotting the slope curve at station O, or $x = L$. We note that the values of the sums of the increments must be divided by EI in order to find the ordinate of the slope curve at any point x.

For the deflection curve we note that the value of the deflection is known at point O. At point O the deflection is zero. We therefore start our deflection curve at O. We notice in this case that the deflection is upward. The ordinate at any point is the summation of the slope up to the point x, but in this case the quantity is not divided by EI.

151. Continuous Spar.—The general procedure in analyzing a continuous wing beam is as follows: The beam loading is determined from the air-loading conditions as specified by the design conditions, or by government agencies, and from this the beam and chord loadings are determined. In order to determine the axial load which occurs in the spars, it is necessary to know the reaction at the supports. This implies a knowledge of the reaction in the flying wires or the anti-flying wires as the case may be. The reactions in the outer supports are usually determined by the application of the ordinary three-moment equation.

The three-moment equation (as may be verified by reference to any standard text on strength of material) is

$$\frac{M_aL_1}{E_1I_1} + \frac{2M_bL_1}{E_1I_1} + \frac{2M_bL_2}{E_2I_2} + \frac{M_cL_2}{E_2I_2} = \frac{w_1L_1^3}{4E_1I_2} + \frac{w_2L_2^3}{4E_2I_2} \quad (261)$$

The left-hand side of the three-moment equation is the same regardless of the type of loading on the bay. The terms on the right-hand side of the equation depend upon the type of loading on the beam. For example, there are terms for concentrated

loads, distributed loads, or the combination of the two types of loading (see handbooks on structures). A term is added for each type of loading in each bay. Now in applying the three-moment equation to the example of Fig. 64, we write the equation for the bays A to B and B to C. From conditions of symmetry the moment at A equals the moment at B, which condition enables us to solve for the values of the moments at A and B with one

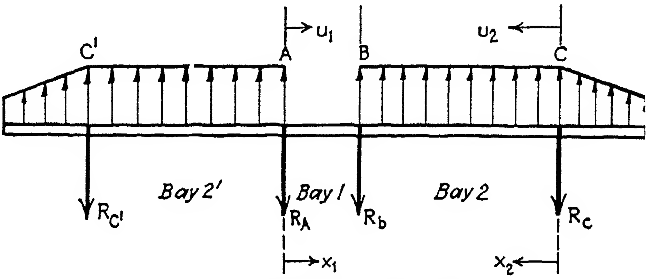


FIG. 64.—Continuous wing beam.

application of the three-moment equation, since the bending moment C is known from the conditions of the overhang and since all the other quantities necessary are known. By taking the summation of the moments about the point B with reference to the portion of the beam to the right of B, we have

$$M_b = -R_C L_2 + \Sigma f(u_2)(L_2 - u_2) \tag{262}$$

from which, since M_b is known, R_c can be found. In most cases the reaction R_c can be determined by ordinary methods of statics, assuming that the bending moment at B is zero and taking the summation of the moment about the point B. Neither one of these methods, however, is exact, inasmuch as the moment at B is influenced considerably by the secondary stress in the beam due to the axial load. However, they make very good approximations.

If the reaction at C is determined, the axial component in the spar may then be determined; and the so-called precise three-moment equation, which involves the quantities pertaining to the beam as a column, may be applied in so far as Hook's law applies. This precise equation will be considered after we have studied the problem of elastic instability (see Chap. 11).

152. Elastic Instability.—The strength of many of the members of an aircraft structure ordinarily does not depend upon the

strength of the material but rather upon the magnitude of the modulus of elasticity of the material; that is, the strength of the members of the structure depends upon whether the members, as, for example, a slender strut, a steel fuselage tube, or a thin sheet of metal in monocoque construction, are stable or unstable. This means that for a long strut, where stability is the criterion of strength, it makes very little difference whether we use a high-grade, high-strength steel or a low-grade, low-strength steel, since the modulus of elasticity of the two is practically the same. Let us give this stability problem more consideration and attempt

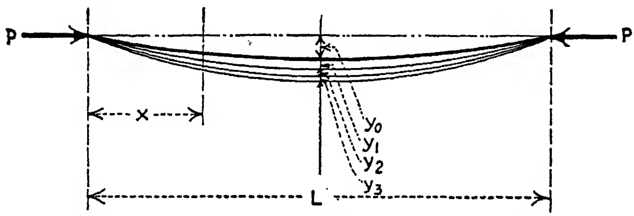


FIG. 65.—Successive increments of deflection of an initially bent strut.

to visualize just why a member is stable or unstable. Ordinarily this problem is presented purely from a mathematical standpoint involving differential equations of the second order, such as the simple column equation

$$EI\frac{d^2y}{dx^2} = -Py \tag{263}$$

Let us attempt to visualize the physical significance of the instability problem. For this purpose let us apply our method of graphical integration of the loading on a beam or its equivalent. Let us assume, for example, in Fig. 65, that we have a beam which is slightly bent, we shall say, by a side load. Let us assume that this beam is subjected to an axial load P, which is applied along the neutral axis of the beam in its unstrained position. Let us assume that the deflection is extremely small in comparison to its length, for example, a length of 60 in. and a deflection of $\frac{1}{10}$ in. Now for all practical purposes we may assume that this deflection is proportional to the sine, that is,

$$y = -y_0 \sin \frac{\pi x}{L} \tag{264}$$

We note in this equation that when $x = L/2$, $y = y_0$ and the deflection is at the center; when $x = 0$, $y = 0$; and when $x = L$, $y = 0$. While the true curve may not be exactly this curve, for the small deflection of such as $1/10$ in. in 60 in., any curve that could be drawn with this maximum deflection could not be distinguished by the eye from any other curve. Now the bending moment at any point x is

$$M_x = Py_0 \sin \frac{\pi x}{L} \tag{265}$$

Thus

$$EI\frac{d^2y}{dx^2} = Py_0 \sin \frac{\pi x}{L} \tag{266}$$

Let us now determine the deflection due to this primary bending moment M_x. We note that there will be an additional deflection due to the primary bending moment. Likewise, a second additional deflection will be caused by the additional secondary bending moment. Let us first, however, determine the additional deflection due to this primary bending moment, which we may designate as the first secondary deflection. We have

$$EI\frac{dy}{dx} = \frac{-PL}{\pi}y_0 \cos \frac{\pi x}{L} + C_1 \tag{267}$$

In order to evaluate the constant C_1, we note that when

$$x = \frac{L}{2}, \qquad \frac{dy}{dx} = 0 \tag{268}$$

We find therefore that

$$C_1 = 0.$$

Integrating again, we have

$$EIy = \frac{-PL^2}{\pi^2}y_0 \sin \frac{\pi x}{L} + C_2 \tag{269}$$

Now, to evaluate the constant C_2, we note that when

$$x = 0, \qquad y = 0 \tag{270}$$

Therefore

$$C_2 = 0$$

We note now that the primary deflection plus the first secondary deflection gives the first approximate deflection:

$$y(\text{approx.}) = -y_0 \sin \frac{\pi x}{L} - \frac{PL^2}{\pi^2 EI} y_0 \sin \frac{\pi x}{L}. \tag{271}$$

which is

$$y \ (\text{approx.}) = -y_0 \sin \frac{\pi x}{L}\left(1 + \frac{L^2 P}{\pi^2 EI}\right) \tag{272}$$

The first secondary bending moment due to this added deflection is

$$M_1 = P\left(\frac{L^2}{\pi^2} \frac{P}{EI} y_0 \sin \frac{\pi x}{L}\right) \tag{273}$$

in which we used a positive sign because the moment is positive though the deflection is negative. Now let us determine from this equation the second secondary deflection by successive integrations. We find that

$$EI\frac{dy}{dx} = -\frac{L^3}{\pi^3} \frac{P^2}{EI} y_0 \cos \frac{\pi x}{L} + C_3 \tag{274}$$

We notice, as previously, in the evaluation of the constant, that $C_3 = 0$; we have, therefore, the second secondary deflection

$$y = -\frac{L^4}{\pi^4} \frac{P^2}{(EI)^2} y_0 \sin \frac{\pi x}{L} + (C_4 = 0) \tag{275}$$

Thus the total deflection y, including the primary deflection, the first secondary deflection, and the second secondary deflection, etc., is

$$y = -y_0 \sin \frac{\pi x}{L}\left(1 + \frac{L^2 P}{\pi^2 EI} + \frac{L^4 P^2}{\pi^4 E^2 I^2} + \cdots\right) \tag{276}$$

If we let

$$\frac{P}{EI} = j^2 \qquad \text{and} \qquad \frac{L^2}{\pi} j^2 = g$$

we have

$$y_0 \sin \frac{\pi x}{L}(1 + g + g^2 + g^3 + \tag{277}$$

which is a geometric series. We note that the geometric series is convergent when g is less than 1. Therefore, the series for the deflection is convergent when

$$\frac{L^2 P}{\pi^2 EI} < 1$$

or

$$P < \frac{\pi^2 EI}{L^2} \tag{278}$$

We recognize equation (278) as Euler's critical loading formula for a column bent in its simplest form. The process which has been performed mathematically would give successive deflections as indicated in Fig. 65.

If the series is divergent, it means that each one of the successive double integrations gives a less value for the deflection than the preceding double integration. This means then that, if the deflections y_1, y_2, y_3, and y_4 are successively smaller and smaller, and if we repeat this integration either mathematically or graphically, we will reach a limit for the point of stability of the strut. However, if our successive double integrations give us increments which are successively greater than the preceding increment, the process may be continued indefinitely, so that the sum of the series would be infinite; consequently under such a condition the strut is unstable.

If we let

$$P_e = \frac{\pi^2 EI}{L^2} \tag{279}$$

in which P_e is the Eulerian critical load, we have

$$M_x = Py_0 \sin \frac{\pi x}{L}\left[1 + \frac{P}{P_e} + \left(\frac{P}{P_e}\right)^2 + \left(\frac{P}{P_e}\right)^3 \cdot \cdot \cdot \right] \tag{280}$$

If P in equation (280) is less than P_e, the series will be convergent. P in this case is the actual applied axial load on the beam. The maximum bending moment will occur at the center of the beam since we have assumed the deflection to be a maximum at this point; hence the maximum stress in the beam will occur also at this point. We have at the center

$$M_0 = Py_0\left[1 + \frac{P}{P_e} + \left(\frac{P}{P_e}\right)^2 + \left(\frac{P}{P_e}\right)^3 + \cdot \cdot \cdot \right] \tag{281}$$

Now we note that the geometrical series may be summed as follows:

$$S = \frac{1}{1 - g} \tag{282}$$

which may be verified by dividing as indicated. We thus have

$$M_t = \frac{Py_0}{1 - \frac{P}{P}} \tag{283}$$

153. Strength of a Strut with Bending Load.—As previously pointed out, the theory of elasticity does not account for the strength of the material in any way. Our calculations here will give us only the critical loading on the structure, *assuming that the material will not break.* It is quite obvious that there are practical limitations to the calculations in that the material may fail before the elastic instability is reached. This point is the essential difference between the theory of elastic instability and the theory of the rupture in a beam. If the fiber stresses are the criterion of the strength of this beam, before the instability theory will apply, the fiber stresses must be below the proportional limit. Assuming then that the fiber stresses are below the proportional limit, let us determine the relationship between the fiber stresses. For this purpose let us express M_o in terms of the fiber stress in the outer fiber of the beam. We have then [see equation (203)],

$$M_o = \frac{f_{bu}I}{} \tag{284}$$

in which f_{bu} is the ultimate allowed fiber stress in bending. Now, as an example, let us assume the deflection y_0 in the beam is caused by a concentrated load at the center of the beam. This type of loading will not produce a sine-curve deflection in the beam. However, for small deflections we probably may assume that the curve is a sine curve. The bending moment at the center of the beam is

$$M_b \quad \frac{WL}{4} \tag{285}$$

and the deflection at the center of the beam due to this bending moment is

$$y_0 = -\frac{WL^3}{48EI} \tag{286}$$

Substituting the M_b from equation (285) in equation (286), we have

$$y_0 = -\frac{M_b L^2}{12EI} \tag{287}$$

We note in equation (287) that

$$\frac{L^2}{EI} = \frac{\pi^2}{P_e} \tag{288}$$

Substituting these values found in equations (284), (287), and (288) in equation (283), and adding M_b we have

$$\frac{f_{bu}I}{c} \quad \frac{\pi^2 P M_b}{12\left(1 - \frac{P}{P_e}\right)P_e} + M_b \tag{289}$$

Now we note that in Euler's formula

$$\frac{P_e}{A} \quad \frac{\pi^2 EI}{L^2 A} \qquad \text{or} \qquad \frac{P_e}{A} = \frac{\pi^2 E}{(L/\rho)^2}$$

where $\rho = \sqrt{I/A}$ = the radius of gyration.
The fiber stress is

$$\frac{P_e}{A} = f_{cu} \tag{290}$$

in which f_{cu} is the ultimate compressive fiber stress of the column.

Now expressing M_b in terms of the fiber stress, we have

$$M_b = \frac{f_{ba}I}{c} \tag{291}$$

in which f_{ba} is the allowable bending stress. Now, substituting the value of M_b from equation (291) in equation (289), we have

$$\frac{f_{bu}I}{c} \quad \frac{\pi^2 \frac{P}{A} f_{ba} I}{12c\left(\frac{P_e}{A} - \frac{P}{A}\right)} + \frac{f_{ba}I}{c} \tag{292}$$

which reduces to

$$f_{bu} = \frac{\pi^2 \frac{P}{A} f_{ba}}{12\left(f_{cu} - \frac{P}{A}\right)} + f_{ba} \tag{293}$$

Equation (298) gives us the relation between the allowable ultimate bending stress in the beam in terms of the allowable ultimate compressive stress. In this equation we set the allowable ultimate bending stress f_{bu} as the proportional limit of the material, and we set the allowable ultimate compressive stress f_{cu} as that which will cause elastic instability, as determined by Euler's critical-load formula or by other appropriate formulas. Now the question arises as to what stress due to the axial load and what stress due to the applied side load will be allowed in the beam.

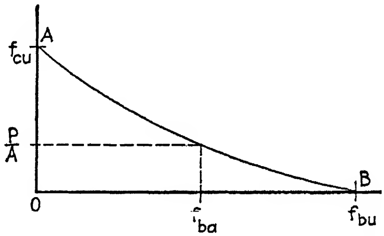


FIG. 66.—Elementary chart for combined loading.

We note in this case that we have not made any allowance in the stress of the material for the axial load which is impressed upon the material. It is quite apparent that the ultimate allowed bending stress includes the P/A stress. Consequently, to make this formula more applicable, the f_{bu} should be replaced by $f_{bu} - \frac{P}{A}$. Substituting this value in equation (293), we have

$$\left(f_{bu} - \frac{P}{A}\right) = \frac{\pi^2 \frac{P}{A} f_{ba}}{12\left(f_{cu} - \frac{P}{A}\right)} + f_{ba} \tag{294}$$

from which, solving for f_{ba}, we find

$$f_{ba} = \frac{12\left(f_{bu} - \frac{P}{A}\right)\left(f_{cu} - \frac{P}{A}\right)}{\pi^2 \frac{P}{A} + 12\left(f_{bu} - \frac{P}{A}\right)} \tag{295}$$

Equation (295) gives us the allowable bending stress in terms of the applied compressive stress and the ultimate stresses in bending and compression. The curve of equation (295) is of the form shown in Fig. 66.

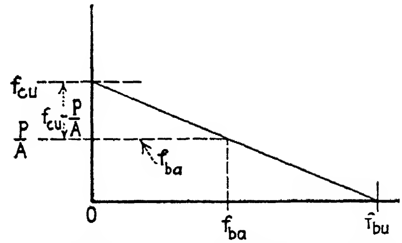


FIG. 67.—Quantities entering into an empirical formula for combined loading.

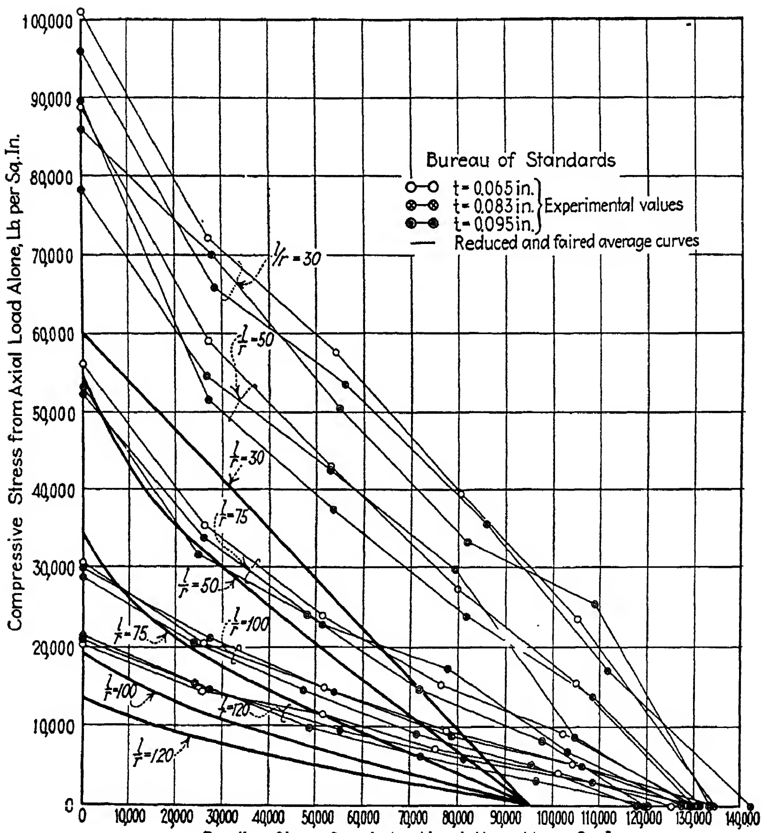


FIG. 68.—Experimental values of bending and compressive stresses. Chromium-molybdenum steel tubes; dia. = 2 in.; thickness of wall from 0.065 in. to 0.095 in.

If AB were a straight line as in Fig. 67, we might write an equation similar to equation (295) as follows: By proportionality of corresponding sides of similar triangles,

$$\left(f_{cu} - \frac{P}{A}\right) : f_{ba} = f_{cu} : f_{bu} \tag{296}$$

or

$$f_{ba} = f_{bu} \frac{\left(f_{cu} - \frac{P}{A}\right)}{f_{cu}} \tag{297}$$

154. Ultimate Strength of Simple Struts with Side Load.— Experiments by the Bureau of Standards showed that the ultimate strength of chrome-molybdenum steel tubes and duralumin tubes, when plotted in the form indicated in Figs. 66 and 67, produced a chart as shown in Figs. 68 and 69.[1]

[1] Figure 68 represents the experimental results obtained on 2-in. chrome-molybdenum steel tubes with wall thicknesses of from 0.065 in. to 0.095 in. in transverse, column, and combined column and transverse tests.

In the transverse tests the load was applied at the third points to a freely supported specimen. In the column test the load was applied axially through the ball-and-socket fixtures to approximate "round-end" conditions.

In the combined tests the above two methods of application of load were used simultaneously.

In the figure, the bending stresses produced by the lateral loads are plotted against the axial compressive stresses produced by the axial loads.

The bending stresses were calculated by the simple beam formula $f_b = Md/2I$ in which M = bending moment, produced by the lateral load, d = outside diameter of the tube, and I = moment of inertia of the cross section.

The axial stress is equal to P/A where P is the maximum axial load carried by the tube under the test conditions and A the cross-sectional area. Each point on these experimental curves represents an average of two determinations.

It will be noticed that for all l/r ratios which were used, the stresses calculated in this manner for the maximum test loads do not show any significant effect of the thickness of wall. Therefore, the results obtained on different thicknesses of wall were averaged for each particular l/r ratio. These average curves are not shown on the diagram.

The average curves meet the axis of abscissas at points which correspond to the maximum bending stresses in the transverse test of from 117,850 to 142,050 lb. per square inch. Although these calculated bending stresses in the transverse test (when the axial load is zero) vary with l/r and thick-

It will be noted in this case, however, that the chart as determined by the Bureau of Standards gives the allowable bending stress or the allowable compressive stress in the strut in terms of the ultimate bending stress, that is, the modulus of rupture, and the ultimate stress as that of a pure column, whether the column be within Euler's range or whether it be a very short column involving the strength of the material. Now, although the theory involved in equation (295) is applicable with very little

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ness, the differences are small, being considerably less than the scatter observed in the combined tests.

In order to combine these results on a safe basis, both the axial and the transverse stresses were reduced in the following manner:

The specifications (Navy Department Specification 10231-*A*, Feb. 7, 1925) call for 95,000 lb. per square inch ultimate and 60,000 lb. per square inch yield point. The average column strength yield of the material was 94,000 lb. per square inch. The average axial stresses calculated from the tests were therefore all reduced in the proportion 60,000/94,000. The transverse stresses were similarly reduced, using the specified value of 95,000 lb. per square inch for the ultimate.

As an illustration of the above procedure, let us consider $l/r = 50$ and the bending stress = 40,000 lb. per square inch. The average experimental curve for $l/r = 50$ intersects the axis of abscissas at 129,900. The abscissas of the given point on the average experimental curve is 40,000 and therefore the abscissa of the corresponding point on the reduced average curve is $40{,}000 \times 95{,}000/129{,}900 = 29{,}250$. The ordinate of the given point on the average experimental curve is 47,000 and therefore the ordinate of the corresponding point on the reduced average curve is

$$47{,}000 \times \frac{60{,}000}{94{,}100} = 30{,}000.$$

Consequently, the reduced average curve passes through the point 29,250 on the abscissas and 30,000 on the ordinate. These reduced average curves were faired and the diagram gives these faired curves. The reduced values for $l/r = 30$ lay so close to a straight line that the faired curve was drawn straight.

The average experimental curves and the reduced average curves (not faired) were omitted for the sake of clearness.

It will be seen that these average curves, reduced on the basis of the specification values for yield point and ultimate, lie below all the experimental points. These reduced curves should therefore furnish a safe basis for calculating the stresses which tubes of material meeting these specifications can carry.

Figure 69 represents the experimental results obtained on 1½-in. duralumin tubes with wall thicknesses of from 0.032 in. to 0.072 in. in transverse, column, and combined column and transverse tests. The same general procedure as outlined above was used.

error up to the elastic limit, yet it cannot be applied to the

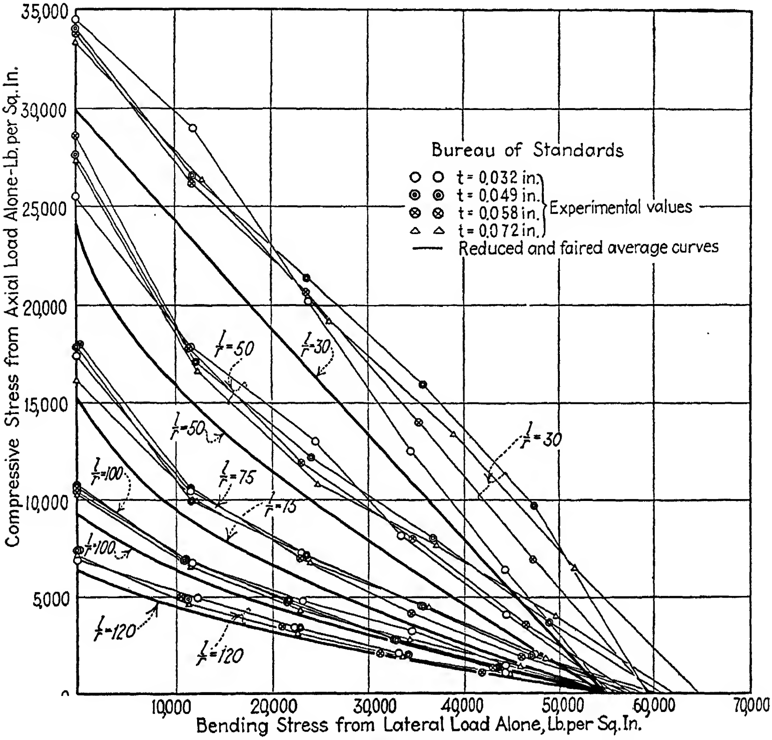

FIG. 69.—Experimental values of bending and compressive stresses. Duralumin tubes; dia. = 1½ in.; thickness from 0.032 in. to 0.072 in.

chart of the Bureau of Standards. *However, it gives us an indication as to the form of the empirical equation which may be derived for expressing the relations in the chart of the Bureau of Standards.*

155. Combined Allowable Stress.—A chart of the form of Figs. 68 and 69 does not give us the combined allowable stress in bending and compression. The chart gives us the allowable bending stress, having assumed the compressive stress, or it gives

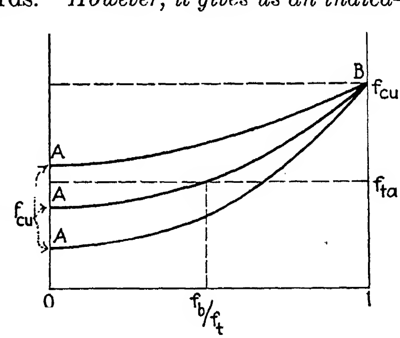

FIG. 70.—Application of chart for axially loaded metal beams.

us the allowable compressive stress, having assumed the bending stress. The more desirable form of representing experimental data on combined loading is as represented in Fig. 70. We have represented in this chart the abscissas as the ratio between the bending stress and the total combined bending and

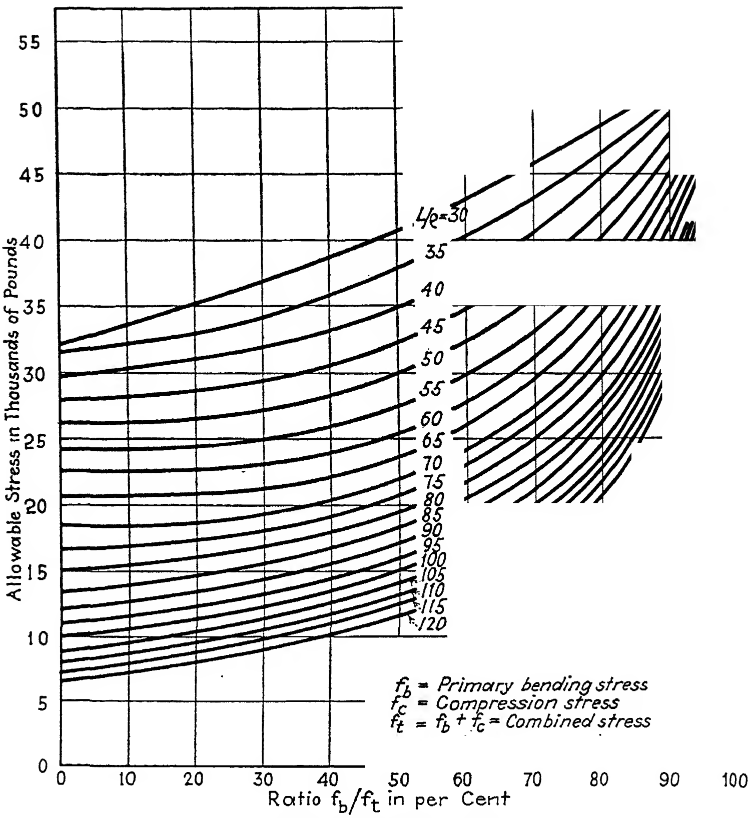

FIG. 71.—Allowable for aluminum-alloy tubes subjected to combined bending and compression.

compressive stresses. The ordinate is the allowable combined total stress. When the bending stress is zero, the beam is a column without a side load. It is represented by the zero abscissa line as *A*, *A*, *A*. The zero abscissa line, therefore, indicates the strength of the column as computed in the range of a long strut by Euler's formula, or in the range of a short strut by an appropriate formula. If no axial load is impressed upon the beam

f_{ta}, the combined allowable stress is equal to f_b. The ratio then is 1. Therefore, the ordinate at the point $f_b/f_t = 1$ is an ordinate of ultimate bending stress. This indicates one point B for any particular material. Thus, for intermediate points, the value of

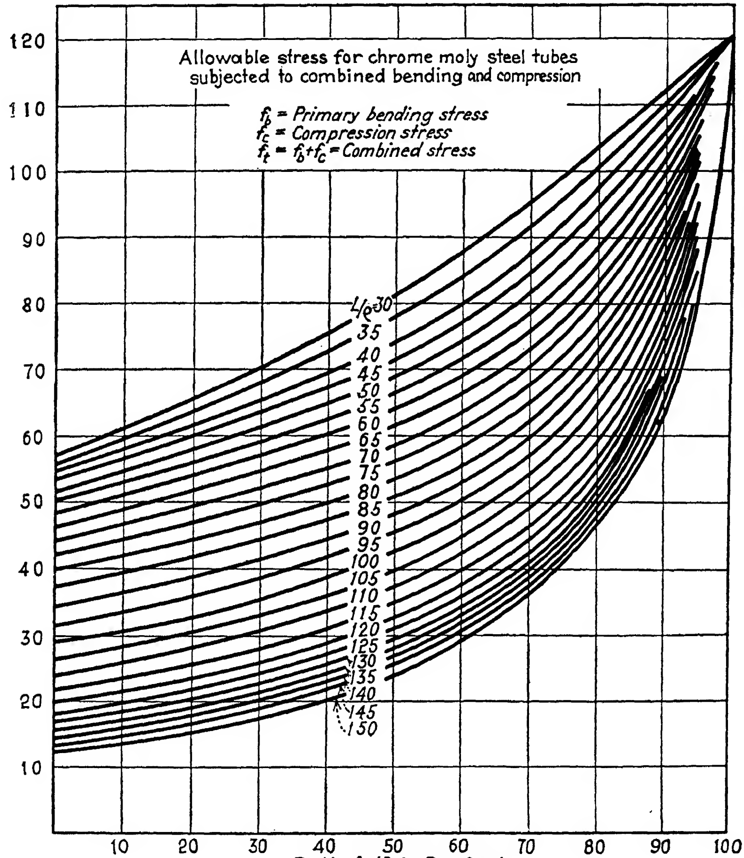

FIG. 72.—Allowable stress for chrome-molybdenum steel tubes subjected to combined bending and compression.

f_{ta} will lie on a curve connecting A and B. Now this type of curve may be determined from Fig. 68 or 69, and it is found to be slightly curved as indicated in the figure. Charts similar to Fig. 70 for aircraft structural materials may be found in the Department of Commerce, Aeronautics Branch, Bulletin 7-A, and in Army and Navy and other handbooks (see Figs. 71, 72,

and 73). It will be noted that these charts do not represent accurately the actual experimental data obtained by the Bureau of Standards. They represent only the stress allowed by law. The allowable experimental stresses are considerably greater than those allowed by the Department of Commerce.

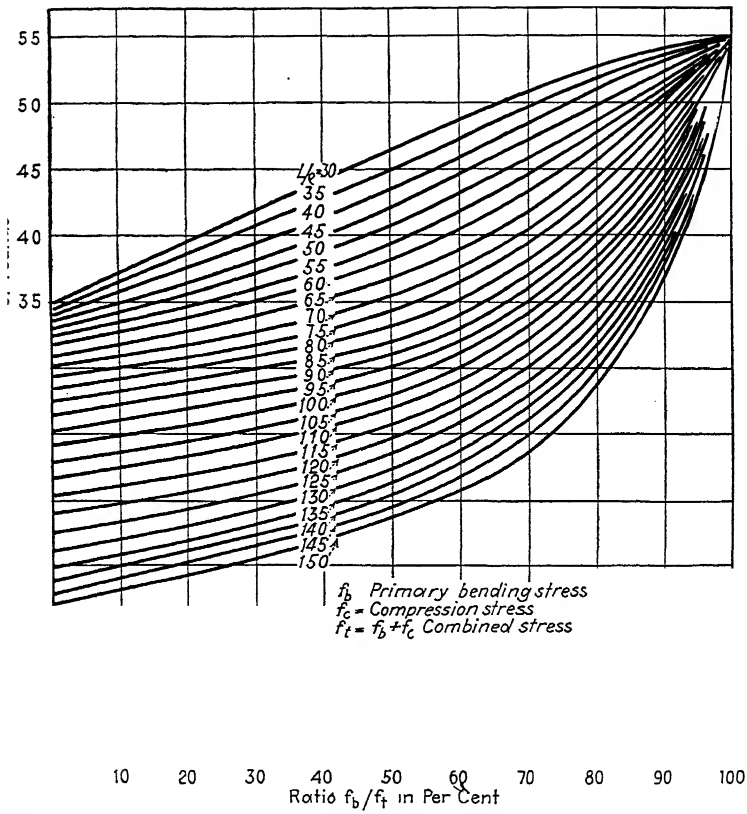

FIG. 73.—Allowable stress for specification 1025 steel tubes subjected to combined bending and compression.

156. Simple Strut.—It should be emphasized that the majority of the structural members in an airplane structure depend for their strength upon their column action, and also that in aircraft work we carry this column action above the theoretical range of calculation, that is, above the proportional limit. Consequently, we are faced with a problem which is very difficult and which has never been solved satisfactorily. Our methods relating to

these conditions, it should be noted, are mostly makeshift methods based upon simple experiments. For the purpose of giving this subject more careful study, let us review the calculations pertaining to the simple strut, that is, the simple column. With reference to Fig. 74*b*, we note that the bending moment at any point x is

$$M_x = -Py \tag{298}$$

We assume a slight deflection in the column. This assumption is justified inasmuch as it is practically impossible to construct a perfectly homogeneous, uniform, straight column. The

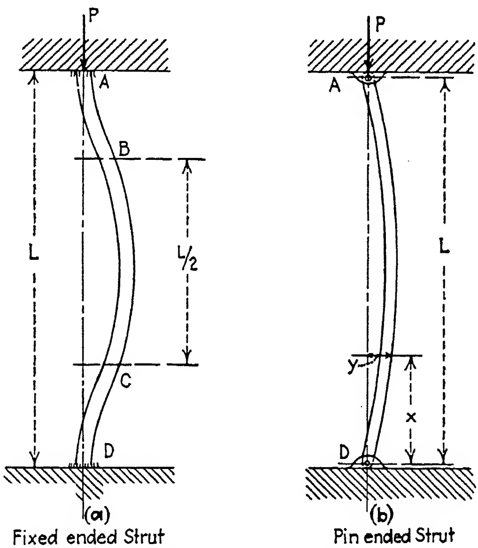

FIG. 74.—Simple struts.

equivalent of the deflection may be found in a slight eccentricity or a slight variation in the material. Writing the deflection equation for the bending moment, we find [see equation (263)] that

$$EI\frac{d^2y}{dx^2} \quad -Py \tag{299}$$

in which we write the negative sign because the deflection is always opposite in sign to the bending moment. We have from this equation

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y \quad 0 \tag{300}$$

Letting

$$\frac{P}{EI} = j^2 \tag{301}$$

and solving the equation, we have

$$y = A \cos jx + B \sin jx \tag{302}$$

In order to evaluate the constants A and B, we note that when

$$x = 0, \qquad y = 0 \tag{303}$$

from which we find that

$$A = 0 \tag{304}$$

We therefore have

$$y = A \sin jx \tag{305}$$

It is therefore apparent that the curve is a sine curve. From the nature of the problem we note that the constant A will be the maximum deflection y_0 at the center of the loop. When $x = L/2$, the slope is zero.

The slope is

$$\frac{dy}{dx} = i = Aj \cos jx \tag{306}$$

Substituting these values, we have

$$0 = Aj \cos j\frac{L}{2} \tag{307}$$

In this case the condition requires that

$$\cos j\frac{L}{2} = 0 \tag{308}$$

This condition is satisfied when

$$j\frac{L}{2} = \frac{n\pi}{2} \tag{309}$$

in which n is an integer. We have, therefore,

$$n\pi \tag{310}$$

or, letting $n^2 = c$, the coefficient of fixity,

$$P = \frac{c\pi^2 EI}{L^2} \tag{311}$$

For the simplest case of bending as represented in the figure, $c = 1$, we have

$$P = \frac{\pi^2 EI}{L^2} \tag{312}$$

Now the fiber stress is

$$\frac{P}{A} = \frac{\pi^2 E(I/A)}{L^2} = \frac{\pi^2 E}{(L/\rho)^2} \tag{313}$$

in which ρ is the radius of gyration of the cross-sectional area of the strut. Plotting this curve as noted in Fig. 75 with P/A, the

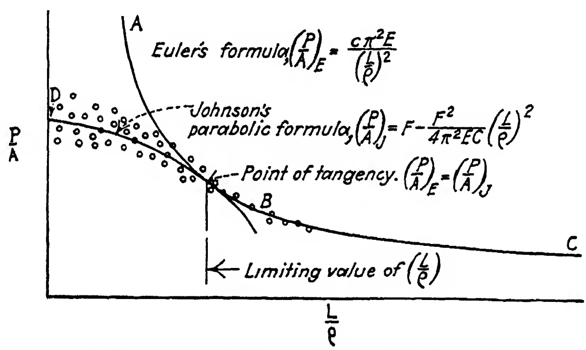

FIG. 75.—Tabulation of data for simple strut.

unit fiber stress, as a function of L/ρ of the strut, we find that Euler's formula gives a curve similar to ABC. It is quite apparent that, if the curve be extended in the direction of A for any distance, the fiber stress of the material will be above the elastic limit and may eventually become infinite; therefore this curve will not be available for use except where the compressive stress in the strut is negligible, that is, in the lower range from B to C. In the range of the short strut (low values of L), it is quite apparent that the ultimate strength of the material determines largely the strength of the strut, so that Euler's theory will not apply. Experiments have shown that there is considerable variation in the experimental results, so that a good general average of these points may be represented by a curve approxi-

mately as BD. There is no theoretical method of determining this curve BD. Empirical formulas have been derived for this, the most important for aircraft use being Johnson's parabolic formula (see Par. 165).

157. Required Practice,* Simple Struts.—Quoting from "Airworthiness Requirements of Air Commerce Regulations for Aircraft," *Aeronautics Bulletin* 7-*A* United States Department of Commerce:

Struts and Columns:

Long, slender struts of metal tubing or spruce shall be designed by the Euler formula,

$$\frac{P}{A} = \frac{c\pi^2 E}{(L/\rho)^2} \qquad \text{or} \qquad P = \frac{c\pi^2 EI}{L^2}$$

Short struts of spruce or steel tubing shall be designed by the Johnson parabolic formula,

$$\frac{P}{A} = f - \frac{f^2L^2}{4c\pi^2E\rho^2} \qquad \text{or} \qquad P = fA - \frac{(fLA)^2}{4c\pi^2EI}$$

Short struts of aluminum-alloy tubing shall be designed by the straight-line formula,

$$\frac{P}{A} = f - \frac{KL}{\rho\sqrt{c}} \qquad \text{or} \qquad P = fA - \frac{KL}{\rho\sqrt{c}}$$

The quantity c in the above formulas is the restraint coefficient which is to be taken as 1 for pin-ended struts and as 4 for truly fixed-ended struts. In the design of a strut its value must be assumed by judgment, but in airplane work, as it is very unlikely that a degree of restraint greater than that represented by $c = 2$ can ever be counted on, that value is the maximum which may be used. The quantity K in the formula for aluminum alloy, an empirically derived constant, is equal to 400; f is the yield-point strength of the material except for aluminum alloy, in which case it is 48,000. In no case shall the value P/A used on aluminum-alloy tubes exceed 40,000 pounds per square inch. Open sections or sections having free edges shall not be designed on the basis of the constants given above for round tubes, but their strength shall be determined by test. Channel sections of aluminum alloy may, in many cases, be designed by use of the constants given in *Air Corps Information Circular* 598. E, I, L, and ρ in the foregoing formulas have their usual significance.

* Frequent changes are made in required practice. The student should obtain the latest bulletin.

It will be noted in these formulas that, when the stress as calculated by Euler's formula reaches the elastic limit, then Johnson's parabolic formula begins to apply. Elaborate sets of nomographic charts are extant for the solution of these equations (see handbooks on airplane structures).

158. Fixity of Joints.—This discussion brings up the question of the fixity of the joints of a strut. We note that in Euler's formula

$$P = \frac{c\pi^2 EI}{L^2}$$

where c is the coefficient of fixity. For example, if the strut is pin ended as in Fig. 74b, c is 1. On the other hand, if the strut has fixed ends as in Fig. 74a, the coefficient of fixity is 4, since this

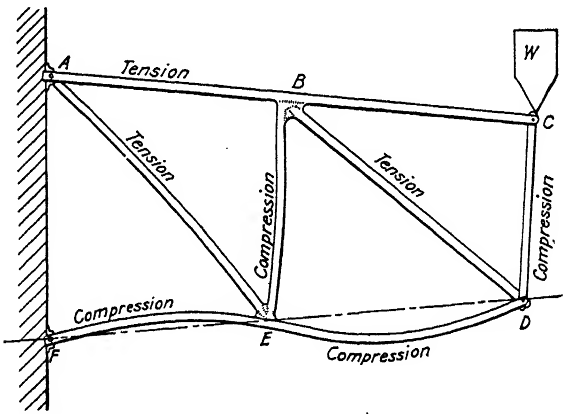

FIG. 76.—Illustrating the unit action of struts in a fuselage.

condition is equivalent to shortening the strut to half the length; this means that the axial load is raised to four times the initial value. Now in an airplane structure, whereas it appears that some of the struts are fixed, they are in reality not fixed, as will be noted from Fig. 76. The struts will take a form somewhat as represented. A joint, as, for example, joint E, rotates as a unit, thus eliminating any possibility of complete fixity. In general, it is found for the usual aircraft structure that the coefficient of fixity should not be selected greater than 2 for welded struts and greater than 1 for pin-ended struts. Under certain conditions a coefficient of fixity of 3 may be allowed, but in general a coefficient of fixity of 4 is never allowed, that is, the structure is never

ideal. This fixity of the ends of struts is a problem which is receiving considerable attention of investigators.

159. Simple Strut with Uniform Lateral Load.—The strut with a side load presents an intricate problem, as we have noted. Let us consider the problem from the standpoint of the theory of

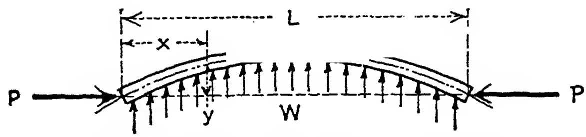

FIG. 77.—Symbols for laterally loaded beam.

elasticity. Consider first the beam with a uniform applied load as in Fig. 77. We note in this figure that the bending moment is

$$M_x = EI\frac{d^2y}{dx^2} = -\frac{WLx}{2} + \frac{Wx^2}{2} - Py \tag{314}$$

Since we desire to find the bending moment particularly, and not the deflection, let us take the second derivative of this equation as follows (assuming that the moment of inertia is constant):

$$\frac{d^2M}{dx^2} + P\frac{d^2y}{dx^2} = w \tag{315}$$

We note that in the second term the second derivative of y with respect to x may be written

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{316}$$

so that we have the equation

$$\frac{d^2M}{dx^2} + \frac{P}{EI}M = w \tag{317}$$

Letting

$$\frac{P}{EI} = j^2$$

we have the solution of this equation as

$$M = A\cos jx + B\sin jx + \frac{w}{j^2} \tag{318}$$

The solution may be verified by substituting in equation (317). In this case, when

$$x = 0, \qquad M = 0$$

from which we find that

$$0 = A + \frac{w}{j^2} \qquad \text{or} \qquad A = \frac{w}{j^2} \tag{319}$$

We also note that when

$$x = L, \qquad M = 0 \tag{320}$$

so that

$$0 = \frac{w}{j^2} \cos jL + B \sin jL + \frac{w}{j^2} \tag{321}$$

from which we find that

$$B = \frac{-\frac{w}{j^2} \cos jL - \frac{w}{j^2}}{\sin jL} \tag{322}$$

160. Lateral Load a Constant Proportion of Axial Load.—Let us now consider another problem in which the lateral load is a fixed proportion of the axial load as, for example, in Fig. 78. Writing the equation for this type of load, we find that

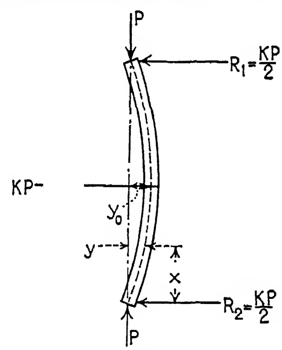

FIG. 78.—Beam loaded laterally with concentrated load.

$$M_x = EI\frac{d^2y}{dx^2} = \frac{KP}{2}x - Py \tag{323}$$

from which we have

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{K}{2}\frac{P}{EI}x \tag{324}$$

The solution gives

$$y = A \cos jx + B \sin jx + ax + bx^3 \tag{325}$$

in which a and b are to be determined by substitution in (324), as

$$\frac{d^2y}{dx^2} = -Aj^2 \cos jx - Bj^2 \sin jx + 6b^2x \tag{326}$$

From (324),

$$\begin{aligned} &-Aj^2 \cos jx - Bj^2 \sin jx + 6b^2x \\ &\quad Aj^2 \cos jx + Bj^2 \sin jx + j^2ax + j^2bx^3 = \frac{K}{2}j^2x \end{aligned} \tag{327}$$

Equating coefficients of like-powered variables, we have

$$6b^2 + j^2 a = \frac{K}{2}j^2$$

$$j^2 b = 0, \qquad b = 0 \tag{328}$$

Thus

$$a = \frac{K}{2}$$

The complete solution is therefore

$$y = A \cos jx + B \sin jx + \frac{K}{2}x \tag{329}$$

We note that when

$$x = 0, \qquad y = 0, \qquad \text{so that} \qquad A = 0 \tag{330}$$

Thus

$$y = B \sin jx + \frac{K}{2}x \tag{331}$$

We have

$$\frac{dy}{dx} = i = \text{slope} = Bj \cos jx + \frac{K}{2} \tag{332}$$

Thus, when $x = L/2$, $i = 0$, if KP is at $x = L/2$,

$$0 = Bj \cos j\frac{L}{2} + \frac{K}{2} \tag{333}$$

so that

$$B = -\frac{K/2}{j \cos \frac{jL}{2}} \tag{334}$$

Therefore

$$y = \frac{-K/2}{j \cos \frac{jL}{2}} \sin jx + \frac{K}{2}x \tag{335}$$

We note

$$M = EI\frac{d^2y}{dx^2} = EI \frac{K/2}{j \cos \frac{jL}{2}} j^2 \sin jx \tag{336}$$

and

$$x = \frac{L}{2}, \qquad M = M_{(\text{max.})} \tag{337}$$

Thus

$$M_{(\text{max.})} = \frac{EIjK}{2 \cos \frac{jL}{2}} \sin \frac{jL}{2} = \frac{EIjK}{2} \tan \frac{jL}{2} \tag{338}$$

or, substituting the value of j, we have

$$M_{(\text{max.})} = \frac{EIK}{2}\sqrt{\frac{P}{EI}} \tan \sqrt{\frac{P}{EI}}\frac{L}{2} \tag{339}$$

Now since

$$f = \frac{P}{A} \pm \frac{Mc}{I} \tag{340}$$

then

$$f_{(\text{max.})} = \frac{P}{A} + \frac{Mc}{I} \tag{341}$$

161. Stress Above the Proportional Limit.—Thus, letting f_t be the total stress, the maximum compressive stress is

$$f_{(\text{max.})} = f_t \quad \sqrt{\frac{P}{EI}}\frac{L}{2} \tag{342}$$

From this equation we can compute the maximum fiber stress. It will be noted in this case that the fiber stress in the beam is proportional to the bending moment and that the deflection is also proportional to the bending moment; therefore, the fiber stress is proportional to the deflection. Now, if an experiment be performed on a strut of this nature in which a set of levers is incorporated so that an applied side load of a given proportion of the axial load may be developed, and the deflection, applied load, and all constants may be taken, we find that we obtain a deflection curve similar to the curve ODC in Fig. 79. It will be noted, however, that the theoretical computation of the deflection y_0 will give a curve similar to OAB. Where the curve OAB departs from the curve ODC, we have the elastic limit of the material. It is apparent, therefore, that the use of the theoretical formula for calculation of the stress for deflection above the

elastic limit is invalid. Since the fiber stress in the beam is proportional to the deflection, as previously noted, it is apparent that the fiber stress above the elastic limit cannot be determined by this theoretical method. If we extend the chart in Fig. 79 to include the bending moment in terms of the deflection, we find that the applied bending moment, which is proportional to the axial load, follows a curve of the nature of *OGH*, while the secondary bending moment Py_0 follows a curve similar to *OEF*, whereas the total bending moment follows a curve similar to *OIK*. It is apparent from these experimental curves that the maximum bending moment will ordinarily not occur within the elastic limit, but will occur at a stress much higher than the elastic limit; this shows us that special consideration will have to

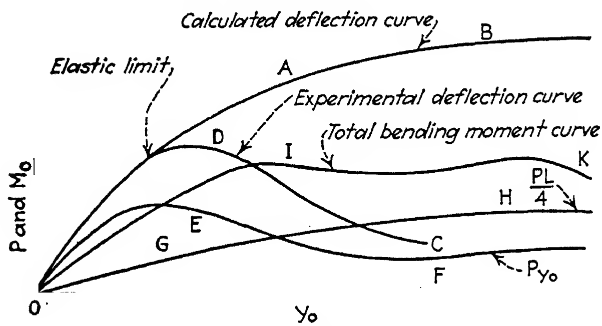

FIG. 79.—Approximate variation of quantities for axially loaded beam above elastic limit.

be given to the calculation of the stress, under these combined loading conditions, above the elastic limit.

162. Struts of Equal Crumpling and Buckling Strength.—It is quite apparent that if we increase the diameter of a tubular strut, while holding the cross-sectional area of the material a constant and thereby making the walls thinner, the strut may fail by crumpling of the thin walls. In Fig. 80 is reproduced the Euler-Johnson curve for a tube. As we decrease the length, the term L/ρ becomes smaller. Now at some point as at E crumpling occurs. It is obvious that any further decrease in L will not give any increase in the strength of the strut; hence the line *EF* represents the strength of such short struts.

The crumpling strength of the thin walls is a function of the ratio of radius r to thickness t of the material (see Sec. IV, Chap. 13, for a further discussion of this problem).

163. Eulerian Strut with Variable Cross Section.—Calculated values of the moment of inertia of the cross-sectional area of several tapered struts outline a curve similar to that shown in Fig. 81*a*.

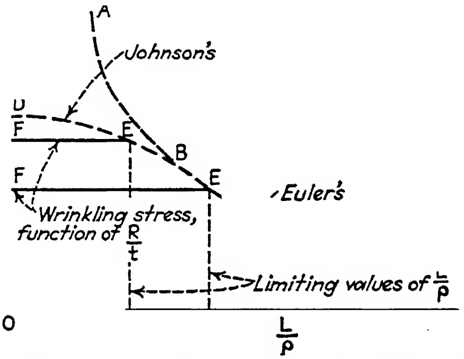

FIG. 80.—Tabulation of data for columns with walls of thin sheet metal.

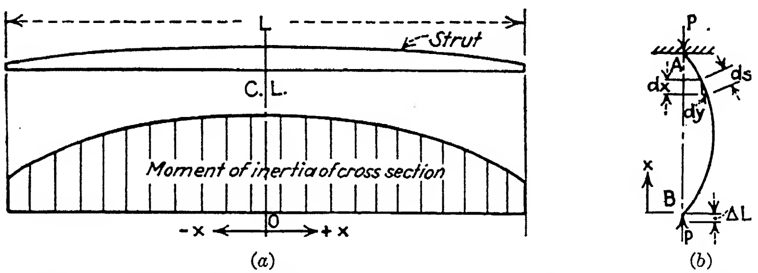

FIG. 81.—(*a*) Strut with variable moment of inertia; (*b*) elements of a bent strut.

This curve is of parabolic form. In the simple strut equation

$$EI\frac{d^2y}{dx^2} + Py = 0 \tag{343}$$

I is a variable, varying along the parabolic curve. The parabola is of the form

$$I = I_0 - I_0\frac{x^2}{[L/2]^2} = I_0\left[1 - \frac{4x^2}{L^2}\right] \tag{344}$$

Note that the origin is taken at the center line of the strut. When $x = 0$, $I = I_0$, and when $x = L/2$, $I = 0$. The end

conditions, while not exact, affect the buckling strength only slightly. Equation (343) becomes

$$EI_0\left[1 - \frac{4x^2}{L^2}\right]\frac{d^2y}{dx^2} + Py = 0 \tag{345}$$

To simplify the symbols, let us replace $2x/L$ by u. We have, therefore,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\frac{2}{L} \tag{346}$$

and

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left[\frac{4}{L^2}\right] \tag{347}$$

Equation (345) becomes

$$EI_0\left[\frac{4}{L^2}\right][1 - u^2]\frac{d^2y}{du^2} + Py = 0 \tag{348}$$

Letting

$$PL^2 \quad = n^2 \tag{349}$$

equation (348) becomes

$$(1 - u^2)\frac{d^2y}{du^2} + n^2y = 0 \tag{350}$$

Let us now assume the solution to be a power series, as

$$y = a_0 + a_1u + a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5 + a_6u^6 + a_7u^7 + \cdots \tag{351}$$

We now substitute the series (351) in equation (350) and solve for the coefficients a_0, a_1, a_2, etc., by means of equating coefficients of like powers of u.

We find

$$\frac{d^2y}{du^2} = 2a_2 + 2 \times 3a_3u + 3 \times 4a_4u^2 + 4 \times 5a_5u^3 + 5 \times 6a_6u^4 + 6 \times 7a_7u^5 + 7 \times 8a_8u^6 + \tag{352}$$

Substituting (352) and (351) in equation (350), we have

$$
\begin{aligned}
&2a_2 + 2 \times 3a_3u + 3 \times 4a_4u^2 + 4 \times 5a_5u^3 + 5 \times 6a_6u^4 \\
&\qquad + 6 \times 7a_7u^5 + 7 \times 8a_8u^6 + \cdots \\
&\quad - 2a_2u^2 - 2 \times 3a_3u^3 - 3 \times 4a_4u^4 - 4 \times 5a_5u^5 \\
&\qquad - 5 \times 6a_6u^6 - 6 \times 7a_7u^5 - 7 \times 8a_8u^6 \cdots \\
&\quad + n^2a_0 + n^2a_1u + n^2a_2u^2 + n^2a_3u^3 + n^2a_4u^4 + n^2a_5u^5 \\
&\qquad + \cdots = 0 \qquad (353)
\end{aligned}
$$

Assuming a_0 and a_1 are the constant of integration, and equating the coefficients of like powers of u, we have:

Coefficients of u^0 $\quad 2a_2 = -n^2a_0, \qquad a_2 = -\frac{n^2a_0}{\ }$

Coefficients of u^1 $\quad 2 \times 3a_3 = -n^2a_1, \qquad a_3 = -\frac{n^2a_1}{2 \times 3}$

Coefficients of u^2 $\quad 3 \times 4a_4 - 2a_2 = -n^2a_2$

$$a_4 = \frac{a_2(2 - n^2)}{3 \times 4} - \frac{n^2a_0}{2 \times 3 \times 4}(n^2 - 2)$$

Coefficients of u^3 $\quad 4 \times 5a_5 - 2 \times 3a_3 = -n^2a_3$

$$a_5 = \frac{a_3(2 \times 3 - n^2)}{4 \times 5} = \frac{n^2a_1(n^2 - 2 \times 3)}{2 \times 3 \times 4 \times 5}$$

Coefficient of u^4 $\quad 5 \times 6a_6 - 3 \times 4a_4 = -n^2a_4$

$$a_6 = \frac{a_4\,(3 \times 4 - n^2)}{5 \times 6} = \frac{-a_0(n^2 - 2)(n^2 - 3 \times 4)n^2}{2 \times 3 \times 4 \times 5 \times 6}$$

Thus y is

$$
\begin{aligned}
&a_0\left[1 - \frac{n^2}{\underline{|2}}u^2 + \frac{n^2(n^2 - 2)u^4}{\ } - \frac{n^2(n^2 - 2)(n^2 - 3 \times 4)u^6}{\underline{|6}} \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \\
&\quad + a_1\left[u - \frac{n^2}{\underline{|3}}u^3 + \frac{n^2(n^2 - 2 \times 3)u^5}{\ } \right. \\
&\qquad\qquad - \frac{n^2(n^2 - 2 \times 3)(n^2 - 4 \times 5)u^7}{\underline{|7}} + \qquad (354)
\end{aligned}
$$

Now when $u = 1$, $y = 0$. Also, when $u = -1$, $y = 0$. Substituting these values of u in turn in equation (354) and adding the two equations, we find that the odd-power series drops out. Thus we have

$$
\begin{aligned}
y = a_0&\left[1 - \frac{n^2}{\underline{|2}}u^2 + \frac{n^2(n^2 - 2)u^4}{\ } - \frac{n^2(n^2 - 2)(n^2 - 3 \times 4)u^3}{\ } \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \qquad (355)
\end{aligned}
$$

If the strut is assumed to fail in the simplest manner, we terminate the series at the third term by letting $n^2 = 2$, in which case

$$y = a_0\left(1 - \frac{n^2}{2}u^2\right) \tag{356}$$

Noting that

$$n^2 = \frac{PL^2}{4EI_0} = 2 \tag{357}$$

we have

$$P = 8\frac{EI_0}{L^2} \tag{358}$$

That is the load required to cause a deflection of this form, hence the critical loading. Noting that for a uniform strut

$$P_e = \frac{\pi^2 EI}{L^2} = \frac{9.86EI}{L^2} \tag{359}$$

Then the ratio is

$$\frac{P_e}{P} = \frac{9.86}{8} \quad 1.23 \tag{360}$$

Thus the nontapered strut is 23 per cent stronger than the tapered strut. The strength-weight ratio is left for the student to calculate.

164. Critical Load of Struts with Variable Cross Sections by Energy Method.—If the point B, Fig. 81*b* moves toward A because of the deflection y of the strut the work done by P is $P\Delta L$. When this work becomes greater than the elastic energy of bending stored in the strut, the strut will be unstable under the load P.

The energy stored in the strut as a result of the bending moment may be expressed as

$$E = \int_0^L \frac{M^2}{2EI}dx \tag{361}$$

Since

$$M = EI\frac{d^2y}{dx^2} \tag{362}$$

we have

$$M = \frac{1}{2}\int_0^L EI\left[\frac{d^2y}{dx^2}\right]^2 dx \tag{363}$$

The work done on the strut by the load P is (see Fig. 81*b*):

$$E = P\Delta L \tag{364}$$

or since

$$\Delta L = \int_0^L (ds - dx) \tag{365}$$

then

$$E = P\int_0^L (ds - dx) \tag{366}$$

Since

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + [dy/dx]^2}dx \tag{367}$$

and since by expansion in a power series we find the value of ds, to infinitesimals of the second order, as

$$ds = \left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]dx \tag{368}$$

we have from equation (366),

$$E = P\int_0^L \left[dx + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx - dx\right] \tag{369}$$

Thus

$$E = \frac{1}{2}P\int_0^L \left(\frac{dy}{dx}\right)^2 dx \tag{370}$$

We therefore have for a stable condition

$$\frac{1}{2}P\int_0^L \left(\frac{dy}{dx}\right)^2 dx < \frac{1}{2}\int_0^L EI\left(\frac{d^2y}{dx^2}\right)^2 dx \tag{371}$$

or

$$P < \frac{\int_0^L EI(d^2y/dx^2)^2dx}{\int_0^L (dy/dx)^2dx} \tag{372}$$

Now let us again consider as an example the tapered strut under the assumption that the curve of deflection is of a parabolic form. Thus

$$I = I_0\left(1 - \frac{x^2}{a^2}\right) \tag{373}$$

where $a = L/2$. Assume the deflection curve,

$$y = y_0\left(1 - \frac{x^2}{a^2}\right) \tag{374}$$

From equation (374),

$$\frac{dy}{dx} = -\frac{2y_0x}{a^2}, \qquad \frac{d^2y}{dx^2} = -\frac{2y}{a^2} \tag{375}$$

Substituting equations (373) to (375) in equation (372), we have, since the origin is taken at the center of the strut,

$$P < \frac{\dfrac{4y}{a^4}\displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(1 - \frac{x^2}{a^2}\right)dx}{\dfrac{4y}{a^4}\displaystyle\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2dx} \tag{376}$$

which becomes

$$P < \frac{EI_0\left(x - \dfrac{x^3}{3a^2}\right)\Bigg|_{-\frac{L}{2}}^{+\frac{L}{2}}}{\left[\dfrac{x^3}{3}\right]_{-\frac{L}{2}}^{+\frac{L}{2}}} \tag{377}$$

Since $a = L/2$,

$$P < \frac{EI_0\left[\dfrac{L}{2} - \dfrac{L}{6} + \dfrac{L}{2} - \dfrac{L}{6}\right]}{\frac{1}{3}[L/2]^3 + \frac{1}{3}[L/2]^3} = \frac{8EI}{L^2} \tag{378}$$

as previously determined.

It may be noted, by trial, that a considerable error in the assumption of the type of curve will produce very little error in the final result.

165. Summary of Design Formulas for Steel and Duralumin Tubes.—Column curves or charts may be used for determining the proper size, but the final allowable load given in the stress analysis should be calculated from the standard formulas given below:

1. *Steel Column:*

Euler's formula for critical load of long struts:

$$\frac{P}{A} = \frac{\pi^2 E}{(L/\rho)^2} \times C = \frac{286{,}000{,}000C}{(L/\rho)^2} \quad \text{for 4130 (chrome-molybdenum) steel} \tag{379}$$

Johnson's parabolic formula for short struts:

$$\frac{P}{A} = F - \frac{F^2}{4\pi^2 EC}\left(\frac{L}{\rho}\right. \tag{380}$$

where

P = maximum allowable load, pounds.
A = area of cross section, square inch.
E = modulus of elasticity, pounds per square inch.
F = yield point of the material, pounds per square inch.
ρ = radius of gyration, inches.
L = length measured between pin centers.
C = fixity coefficient of strut.

Equating equation (379) to (380) and solving for L/ρ, we find the limiting value of $L/\rho = \pi\sqrt{2E/F} \times \sqrt{C}$, or, when $P/A = F/2$, we obtain the following table:

Yield point	Johnson's formula	Limiting $\frac{L}{\rho}$
60,000	$\frac{P}{A} = 60000 - \frac{3.144}{C}\frac{L^2}{\rho^2}$	$98\sqrt{C}$
105,000	$\frac{P}{A} = 105000 - \frac{9.630}{C}\frac{L^2}{\rho^2}$	$74\sqrt{C}$
125,000	$\frac{P}{A} = 125000 - \frac{13.65}{C}\frac{L^2}{\rho^2}$	$68\sqrt{C}$
150,000	$\frac{P}{A} = 150000 - \frac{19.65}{C}\frac{L^2}{\rho^2}$	$62\sqrt{C}$

E = 29,000,000 lb. per square inch for 4130 steel.

In case the calculated P/A from the short-column formulas is greater than the crippling stress of the material, the latter should

be used for the maximum allowable stress. This value should either be determined by test or a conservative assumption should be made.

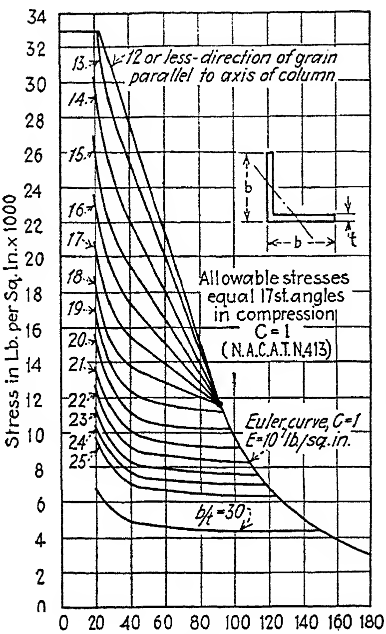

FIG. 82.

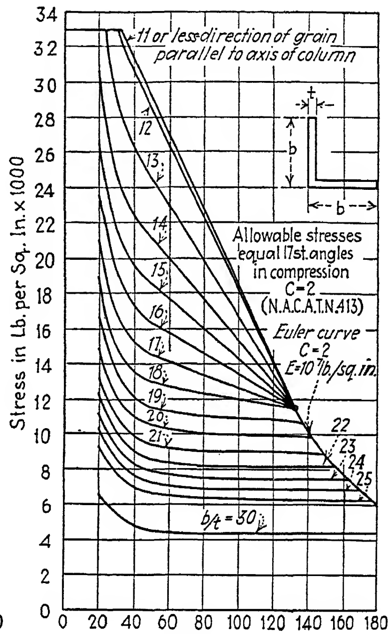

FIG 83.

FIG. 82.—Allowable stresses equal 17st angles in compression for $C = 1$.
FIG. 83.—Allowable stresses equal 17st angles in compression for $C = 2$.

2. *Aluminum-alloy Column:*

E = 10,400,000 lb. per square inch for drawn tubes

$$\text{Euler's formula: } \frac{P}{A} = \frac{102{,}600{,}000\,C}{(L/\rho)^2} \qquad (381)$$

$$\text{Straight-line formula: } \frac{P}{A} \quad 48{,}000 - \frac{400}{\sqrt{C}}\left(\frac{L}{\rho}\right) \qquad (382)$$

Limiting value of $L/\rho = 80 \times C$

Limiting value of P/A = 16,000 lb. per square inch for all values of C

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CHAPTER XI

CONTINUOUS BEAM COLUMNS

166. Methods of Analysis.—The mathematical analysis of the beam column becomes quite complex if one considers special types of loading and variation in the moment of inertia of the cross-sectional area of the beam. For this reason, special methods of analysis from time to time appear in the literature on the subject. These methods are always limited in their scope, and quite often are more difficult to master than the basic theory. For this reason it seems most expedient to present here as much

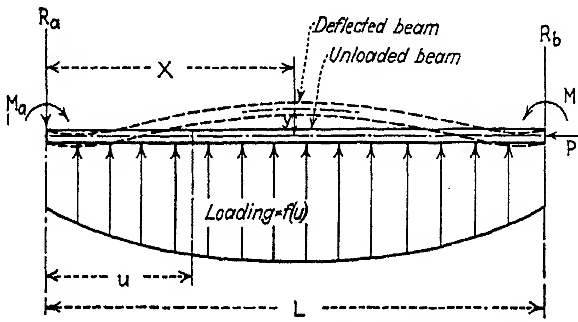

FIG. 84.—Non-uniformly loaded beam column.

of the basic theory as appears practical and to leave the special methods for independent study.

Experiments show that the basic theory is fairly accurate for stresses below the proportional limit of the material. Since the theory is based on Young's modulus of elasticity of the material, the result is subject to the variability of this constant, which is approximately 5 to 15 per cent.

167. Basic Equation for Distributed Load.—With reference to Fig. 84, the bending moment at x for a distributed load $f(u)$ and an axial load P is [see equation (248)],

$$M_x = M_a + \frac{(M_b - M_a)x}{L} - \frac{x}{L}\int_0^L f(u)(L-u)du + \int_0^x f(u)(x-u)du - Py \quad (383)$$

It is desired to find M_x in terms of x (eliminating y). Differentiating equation (383) with respect to x, we have the shear,

$$\frac{dM_x}{dx} = \frac{M_b - M_a}{L} - \frac{1}{L}\int_0^L f(u)(L-u)du + \int_0^x f(u)du - P\frac{dy}{dx} \tag{384}$$

Differentiating again, we have the loading,

$$\frac{d^2M_x}{dx^2} = f(x) - P\frac{d^2y}{dx^2} \tag{385}$$

Since

$$\frac{d^2y}{dx^2} = \frac{M_x}{EI} \tag{386}$$

equation (385) becomes, letting $P/EI = j^2$,

$$\frac{d^2M_x}{dx^2} + j^2M_x = f(x) \tag{387}$$

168. Solution of the Differential Equation.—The solution of this equation is obtained as follows: Multiply the equation by the integrating factor cos jx and obtain the exact differential equation

$$\cos jx\frac{d^2M_x}{dx^2} + j^2M_x \cos jx = f(x) \cos jx \tag{388}$$

the integral of which is

$$\cos jx\frac{dM_x}{dx} + jM_x \sin jx = \int f(x) \cos jxdx + B \tag{389}$$

Now multiply equation (387) by the integrating factor sin jx, and integrate; thus

$$\sin jx\frac{dM_x}{dx} - jM_x \cos jx = \int f(x) \sin jxdx - A \tag{390}$$

Eliminating dM_x/dx between equation (389) and (390), we obtain

$$M_x = A \cos jx + B \sin jx + \frac{\sin jx}{j}\int f(x) \cos jxdx + \frac{\cos jx}{j}\int f(x) \sin jxdx \tag{391}$$

The equation may be written, as may be easily verified,

$$M_x = A \cos x + B \sin jx + \frac{1}{j}\int_0^x f(u) \sin j(x-u)du \tag{392}$$

169. Evaluating the Constants.—The boundary conditions for evaluating the constants are:

When $x = 0, \qquad M_x = M_a$
and when $x = L, \qquad M_x = M_b$
from which

$$A = M_a \tag{393}$$

and

$$B = \frac{M_b}{\sin jL} - \frac{M_a \cos jL}{\sin jL} - \frac{1}{j \sin jL}\int_0^L f(u) \sin j(L - u)du \tag{394}$$

We thus obtain the equation for the moment between the supports

$$M_x = M_a\frac{\sin j(L - x)}{\sin jL} + M_b\frac{\sin jx}{\sin jL} - \frac{\sin jx}{j \sin jL}\int_0^L f(u) \sin j(L - u)du$$
$$+ \frac{1}{j}\int_0^x f(u) \sin j(x - u)du \tag{395}$$

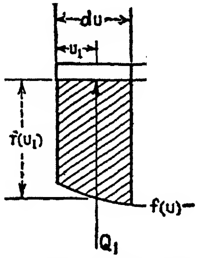

FIG. 85.—Equivalent concentrated load.

170. Simple Beam Column with Uniform Load.—If, in equation (395), we let $M_a = M_b = 0$ and $f(u) = w$, we obtain

$$-jM_x = \frac{w}{j \sin jL}[\sin jx + \sin j(L - x)] - \frac{w}{j} \tag{396}$$

The moment of instability occurs when $\sin jL = 0$. Thus

$$jL \quad m\pi, \qquad \text{or} \qquad \frac{P}{EI} = \frac{m^2\pi^2}{L^2}$$

or

$$\frac{m^2\pi^2EI}{L^2} \tag{397}$$

The beam will probably fail from overstrain of fibers before this load is reached.

171. Simple Beam Column with Concentrated Loads.—If in equation (395) we let $M_a = M_b = 0$ and $f(u)du = Q$, we obtain an equation for concentrated loads Q_1, Q_2, etc. A study of Fig. 85 will show the transformation involved. $f(u)$ is the average magnitude of loading, and du is the distance along the beam

under consideration. For example, $f(u)$ may be 10 lb. per inch and du taken as 2 in. Thus $Q = 10 \times 2 = 20$ lb., considered as a concentrated load. Thus we may write equation (395):

$$M_x = -\frac{\sin jx}{j \sin jL}\left[\sum_0^L Q_1 \sin j(L - u_1) + Q_2 \sin j(L - u_2) + \right.$$

$$\left. + \frac{1}{j}\left[\sum_0^x Q_1 \sin j(x - u_1) + Q_2 \sin j(x - u_2) + \cdots \right]\right] \quad (398)$$

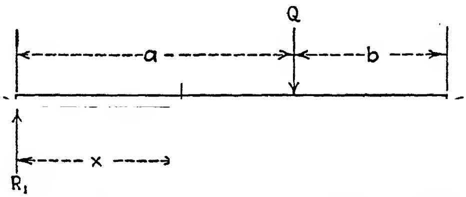

FIG. 86.—Beam column with concentrated lateral load.

For example, the equation for the loading shown in Fig. 86 is

$$M_x = -\frac{\sin jx}{j \sin jL}[- Q \sin jb] + \frac{1}{j}[-Q \sin j(x - a)] \quad (399)$$

If we let $b = x = a = L/2$

$$M_x = +\frac{\sin j\frac{L}{2}}{j \sin jL} Q \sin j\frac{L}{2} = \frac{Q}{2j}\frac{\sin j\frac{L}{2}}{\cos j\frac{L}{2}} \quad (400)$$

If we write KP for Q and $\sqrt{P/EI}$ for j, equation (400) becomes

$$M_{(\text{max.})} = \frac{EIK}{2}\sqrt{\frac{P}{EI}} \tan \sqrt{\frac{P}{EI}}\frac{L}{2} \quad (401)$$

which checks equation (339), derived by another method.

172. Uniform Load on Portion of Beam.—With reference to Fig. 87, we write for equation (395), $M_a = M_b = 0$ and

$$M_x = -\frac{\sin jx}{j \sin jL}\left[\int_0^a f(u) \sin j(L - u)du + \int_a^b f(u) \sin j(L - u)du + \int_b^L f(u) \sin j(L - u)du\right]$$

$$+ \frac{1}{j}\left[\int_0^a f(u) \sin j(x - u)du + \int_a^b f(u) \sin j(x - u)du + \int_b^x f(u) \sin j(x - u)du\right] \quad (402)$$

Since $f(u) = 0$ from O to a, b to x, and from b to L, the integrals with those limits are zero. Thus equation (402) becomes

$$M_x = -\frac{\sin jx}{j \sin jL}\int_a^b (-w) \sin j(L - u)du + \frac{1}{j}\int_a^b (-w) \sin j(L - u)du \quad (403)$$

If x lies between $x = a$ and $x = b$, the limit of the second integral in equation (402), in the second bracket is from a to x, and the third integral does not exist.

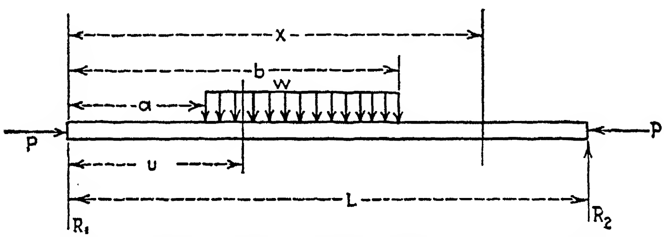

FIG. 87.—Beam column with discontinuous uniform load.

173. Concentrated and Uniform Loads.—An equation for a combination of concentrated and uniform loads may be obtained by adding equations (398) and (402). The student should note particularly, however, that the last integral of the basic equation (395) is valid only from O to x; hence the validity holds also for the special cases of equations (398) and (402).

174. Other Types of Loading.—If the loading curve can be expressed mathematically, its value may be substituted for $f(u)$ in equation (395), and the integrals evaluated. If the loading curve cannot be expressed mathematically, equation (398) may be used. In this case Q is $f(u)\Delta u$, in which $f(u)$ is the mean loading for the interval Δu. Then u is the abscissa of the mean loading.

175. Location of Maximum Moment.—To find the location of the maximum moment, we differentiate equation (395), equate to zero, and solve for x. Thus

$$\frac{dM_x}{dx} = -M_a\frac{j\cos j(L-x)}{\sin jL} + M_b\frac{j\cos jx}{\sin jL} - \frac{\cos jx}{\sin jL}\int_0^L f(u)\ \text{si}$$

$$j(L-u)du + \int_0^x f(u)\cos j(x-u)du = 0 \quad (404)$$

This is the equation for the shear.

In general, the evaluation of x in equation (404) is not possible. In such cases, it is necessary to plot the curve to find the maximum moment.

176. Maximum Moment for Uniform Loading.—Substituting w for $f(u)$ in equation (404), we find

$$\tan jx = \frac{M_b - \frac{w}{j^2} - M_a\cos jL + \frac{w}{j^2}\cos jL}{\left(M_a - \frac{w}{j^2}\right)\sin jL} \quad (405)$$

From this x may be evaluated.

If this value of x be substituted in equation (395), we obtain

$$M_{x(\max.)} = \frac{M_a - \frac{w}{j^2}}{\cos jx} + \frac{w}{j^2} \quad (406)$$

There may not be a maximum between the supports A and B, in which case x will be a negative value or a value greater than L. Equations (405) and (406) may be more readily obtained by the indicated operation on the equations in Par. 159.

177. Variable Moment of Inertia.—If the moment of inertia is a variable, then j in equation (387) is a function of x. This equation has not been solved. If M_a and M_b are known, however, the bending moment, shear, deflection, etc., may be obtained by successive approximate integrations, the principles of which are illustrated in Pars. 147–149, and 152.

If, however, M_a and M_b are functions of j, as in the case of a beam continuous over more than two supports, this method cannot be used, since M_a and M_b cannot be determined. *If the variation of I is small, the average value of I will probably give results sufficiently accurate.* If the minimum value of I is taken, the results will be conservative. Since there is considerable variation in the modulus of elasticity, and since wing beams are the major members of an airplane, it appears that the conservative method should be required.

178. Slope and Deflection of Beam Column.—In equation (395), by substituting $EI\frac{d^2y}{dx^2}$ for M_x and integrating, we obtain

$$EI\frac{dy}{dx} = M_a\frac{\cos j(L - x)}{j \sin jL} - M_b\frac{\cos jx}{j \sin jL} + \frac{\cos jx}{j \sin jL}\int_0^L f(u) \sin j(L - u)du - \frac{1}{j^2}\int_0^x f(u) \cos j(x - u)du + \frac{1}{j^2}\int_0^x f(u)du + C_1 \quad (407)$$

The boundary conditions are:

When $x = L$, $dy/dx = i_b =$ slope at B.

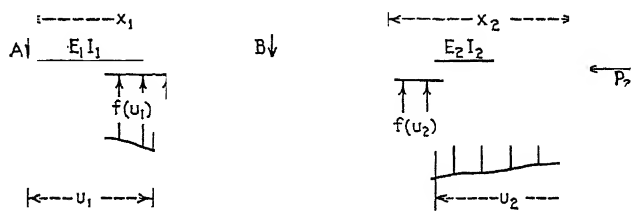

FIG. 88.—Continuous beam column with non-uniform lateral load.

Substituting the boundary conditions in equation (407), solving for C_1, substituting C_1 in equation (407), and integrating, we obtain

$$EIy = -\frac{M_a \sin j(L - x)}{j^2 \sin jL} - \frac{M_a x}{j \sin jL} - \frac{M_b \sin jx}{j^2 \sin jL} + \frac{M_b x \cos jL}{j \sin jL} + \frac{\sin jx}{j^3 \sin jL} - \int_0^L f(u) \sin j(L - u)du - \frac{x \cos jL}{j^2 \sin jL}\int_0^L f(u) \sin j(L - u)du - \frac{1}{j^3}\int_0^x f(u) \sin j(x - u)du + \frac{x}{j^2}\int_0^L f(u) \cos j(L - u)du - \frac{x}{j^2}\int_0^L f(u)du + \frac{x}{j^2}\int_0^x f(u)du - \frac{1}{j^2}\int_0^x uf(u)du + EIi_b + C_2. \quad (408)$$

The boundary conditions are:

When $x = 0$, $y = y_a$. Thus $C_2 = EIy_a + \dfrac{M_a}{j^2}$ (409)

179. Three-moment Equation for Beam Column.—Solving for i_b in equation (408), letting $x = L$, and introducing the subscript 1 for bay 1 of Fig. 88, we obtain

$$i_b = \frac{1}{E_1I_1L_1}\left[\frac{M_aL_1}{j_1 \sin j_1L_1} - \frac{M_a}{j_1^2} + \frac{M_b}{j_1^2} - \frac{M_bL_1 \cos j_1L_1}{j_1^2 \sin j_1L_1}\right.$$

$$- \frac{1}{j_1^3}\int_0^L f(u_1) \sin j_1(L_1 - u_1)du_1 + \frac{L_1 \cos j_1L_1}{j_1^2 \sin j_1L_1}\int_0^L f(u)_1 \sin j_1(L_1 - u_1)du_1$$

$$+ \frac{1}{j_1^3}\int_0^L f(u_1) \sin j_1(L_1 - u_1)du_1 - \frac{L_1}{j_1^2}\int_0^{L} f(u_1) \cos j_1(L_1 - u_1)du_1$$

$$+ \frac{1}{j^2}\int_0^L u_1f(u_1)du_1 - E_1I_1y_a \tag{410}$$

An equation similar to equation (410) is required for the section of the beam from C to B. The only changes in equation (410) necessary are that M_c be substituted for M_a, $-i_b$ for i_b, and y_c for y_a, and that subscript 2 be introduced.

Obtaining this equation for bay 2, adding it to equation (410), simplifying, multiplying by 6, and substituting the notation x for u, we obtain

$$\frac{M_aL_1}{E_1I_1}\alpha_1 + 2\frac{M_bL_1}{E_1I_1}\beta_1 + 2\frac{M_bL_2}{E_2I_2}\beta_2 + \frac{M_cL_2}{E_2I_2}\alpha_2 = \frac{L_1^2}{E_1I_1}\psi_1 + \frac{L_2^2}{E_2I_2}\psi_2 + \frac{6y_a}{L_1} + \frac{6y_c}{L_2} \tag{411}$$

in which

$$\alpha = \frac{6(jL \operatorname{cosec} jL - 1)}{(jL)^2} \tag{412}$$

$$\beta \quad \frac{3(1 - jL \cot jL)}{(jL)^2} \tag{413}$$

and

$$\psi = \int_0^L\left[\frac{6 \sin jL}{(jL)^2 \sin jL} - \frac{6x}{L(jL)^2}\right]f(x)dx \tag{414}$$

180. Three-moment Equation for Uniform Load.—When $f(x) = w$, the term $\frac{L^2}{EI}\psi$ becomes

$$\frac{wL^3}{4EI}\gamma \tag{415}$$

in which

$$\gamma = \frac{3\left(\tan \frac{jL}{2} - \frac{jL}{2}\right)}{(jL/2)^3} \tag{416}$$

181. Functions, α, β, and γ.—The functions, α, β, and γ have been calculated and tabulated (see Table XVII). In some of these tables j as used in this text must be replaced by $1/j$. After the constants α, β, and γ are obtained, equation (411) is used exactly as the ordinary three-moment equation.

182. The ψ-function.—In equation (414) let $x = KL$, where K is a fraction varying from 0 to 1. We have

$$dx = LdK \qquad \text{and} \qquad f(x) = f(KL)$$

When $x = L$, $K = 1$, and when $x = 0$, $K = 0$; thus equation (414) becomes

$$\psi = \int_0^1 \left[\frac{6 \sin KjL}{(jL)^2 \sin jL} - \frac{6K}{(jL)^2}\right] f(KL)LdK \tag{417}$$

LdK is an element of length along the span, and $f(KL)$ is the average intensity of loading at this element. Thus $f(KL)LdK$ is the total loading which may be assumed concentrated at the center of the element. Let this be Q_k; then

$$= \sum_{K=0}^{K=1} \left[\frac{6 \sin KjL}{(jL)^2 \sin jL} - \frac{6K}{(jL)^2}\right] Q_k \tag{418}$$

If we let

$$\frac{6 \sin KjL}{(jL)^2 \sin jL} - \frac{6K}{(jL)^2} = Y_k \tag{419}$$

then

$$\psi = \sum_{(K=0)}^{(K=1)} Y_k Q_k \tag{420}$$

For example,

$$\psi = (YQ)_{(k=0.1)} + (YQ)_{(k=0.2)} + \cdot \cdot \cdot \ (YQ)_{(k=0.9)} \tag{421}$$

The domain of the Y-function is sufficiently limited so that a table of the numerical values may be constructed. Such a table may be found in Reference 6. This table enables one to calculate ψ for any type of loading, so that the function may be incorporated in the three-moment equation.

If the loading curve may be expressed mathematically, that is, if $f(x)$ is a known function of x, ψ may be evaluated by integrating

equation (414). If the loads are concentrated, ψ may be found from equation (418).

TABLE XVII.—CONTINUOUS BEAM COLUMN, THREE-MOMENT EQUATION FUNCTIONS α, β, AND γ[1]
(Abridged for classroom use)

JL, radians	α	β	γ	JL, radians	α	β	γ
1.00	1.1304	1.0737	1.1133	2.70	4.3766	3.7619	3.7863
1.10	1.1617	1.0912	1.1379	2.71	4.4757	2.8121	3.8671
1.20	1.1979	1.1114	1.1686	2.72	4.5795	2.8648	3.9517
1.30	1.2396	1.1345	1.2039	2.73	4.6885	2.9199	4.0405
1.40	1.2878	1.1610	1.2445	2.74	4.8029	2.9778	4.1337
1.50	1.3434	1.1915	1.2914	2.75	4.9233	3.0386	4.2317
1.60	1.4078	1.2266	1.3455	2.76	5.0499	3.1027	4.3349
1.70	1.4830	1.2673	1.4085	2.77	5.1835	3.1702	4.4436
1.80	1.5710	1.3147	1.4821	2.78	5.3245	3.2414	4.5584
1.90	1.6750	1.3704	1.5689	2.79	5.4736	3.3166	4.6797
2.00	1.7993	1.4365	1.6722	2.80	5.6315	3.3963	4.8082
2.10	1.9493	1.5158	1.7967	2.81	5.7990	3.4807	4.9444
2.20	2.1336	1.6124	1.9491	2.82	5.9770	3.5704	5.0892
2.30	2.3640	1.7325	2.1392	2.83	6.1664	3.6659	5.2432
2.40	2.6596	1.8854	2.3822	2.84	6.3685	3.7676	5.4075
2.41	2.6935	1.9031	2.4103	2.85	6.5845	3.8764	5.5832
2.42	2.7287	1.9212	2.4391	2.86	6.8160	3.9928	5.7713
2.43	2.7649	1.9398	2.4687	2.87	7.0646	4.1179	5.9733
2.44	2.8021	1.9589	2.4993	2.88	7.3322	4.2525	6.1907
2.45	2.8403	1.9786	2.5306	2.89	7.6212	4.3977	6.4255
2.46	2.8798	1.9989	2.5630	2.90	7.9343	4.5550	6.6798
2.47	2.9204	2.0198	2.5964	2.91	8.2745	4.7259	6.9561
2.48	2.9624	2.0413	2.6307	2.92	8.6455	4.9121	7.2573
2.49	3.0056	2.0635	2.6662	2.93	9.0516	5.1160	7.5871
2.50	3.0502	2.0864	2.7027	2.94	9.4982	5.3401	7.9496
2.51	3.0963	2.1100	2.7405	2.95	9.9915	5.5875	8.3500
2.52	3.1438	2.1343	2.7794	2.96	10.5393	5.8622	8.7946
2.53	3.1931	2.1595	2.8197	2.97	11.1510	6.1688	9.2910
2.54	3.2437	2.1855	2.8612	2.98	11.8386	6.5134	9.8489
2.55	3.2963	2.2124	2.9043	2.99	12.6171	6.9035	10.4804
2.56	3.3508	2.2402	2.9488	3.00	13.5057	7.3486	11.2013
2.57	3.4072	2.2690	2.9949	3.01	14.5295	7.6813	12.0317
2.58	3.4657	2.2988	3.0427	3.02	15.7219	8.4583	12.9988
2.59	3.5262	2.3297	3.0922	3.03	17.1282	9.1623	14.1393
2.60	3.5890	2.3618	3.1435	3.04	18.8116	10.0049	15.5044
2.61	3.6542	2.3950	3.1968	3.05	20.8620	11.0314	17.1677
2.62	3.7220	2.4295	3.2522	3.06	23.4176	12.3096	19.2388
2.63	3.7925	2.4654	3.3097	3.07	26.6860	13.9446	21.8886
2.64	3.8569	2.5027	3.3696	3.08	31.0160	16.1105	25.3989
2.65	3.9421	2.5415	3.4319	3.09	37.0244	19.1156	30.2701
2.66	4.0218	2.5819	3.4969	3.10	45.9234	23.5659	37.4839
2.67	4.1047	2.6241	3.5646	3.11	60.4566	30.8334	49.2647
2.68	4.1914	2.6680	3.6353	3.12	88.4522	44.8321	71.9577
2.69	4.2820	2.7140	3.7092	3.13	164.7487	82.9812	133.8017

[1] See equations (412), (413), and (416). From Reference 4, by permission.

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CHAPTER XII

PROBLEMS IN TORSION

183. Basic Theory of Torsion.—Three phases of the torsion problem are evident, namely:

1. Torsion involving stresses below the elastic limit.
2. Torsion involving stresses above the elastic limit.
3. Torsion involving the elastic stability of the structure.

The first phase has for its basic theory the mathematical theory of elasticity. The second phase is not subject, at the present state of knowledge, to calculations based on pure theory. Experimental curves, empirical formulas, and approximations

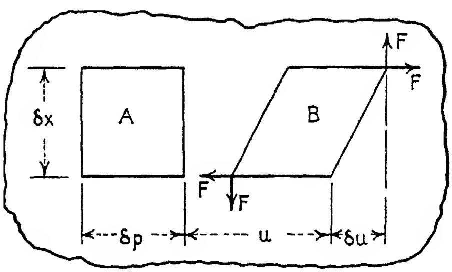


FIG. ! -Elements of shear in a plate.

based on the theory of elasticity form the basis of practical calculations. The third phase is within the field of the theory of elasticity, but, in general, the mathematics involved is too difficult for practical use, unless simplifying assumptions are made which limit the accuracy of the method. Structures of thin sheet metal, in general, are considered in the third classification.

184. Shear Modulus of Elasticity.—With reference to Fig. 89, an elementary block of volume $(\delta x)(\delta p)t$ in an elastic structure subjected to shear is moved from A to B and distorted, as noted, by the shear force. Letting E_s represent the shear modulus, we have

$$E_s = \frac{\text{unit shear}}{\text{unit strain}} \tag{422}$$

$$\text{Unit shear } s = \frac{F}{t(\delta p)}, \qquad \text{and unit strain} = \frac{\delta u}{\delta x} \tag{423}$$

But
$$\delta u = \frac{\partial u}{\partial x}\delta x,$$

so that

$$E_s = \frac{s}{\partial u/\partial x} = s\frac{\partial x}{\partial u} \tag{424}$$

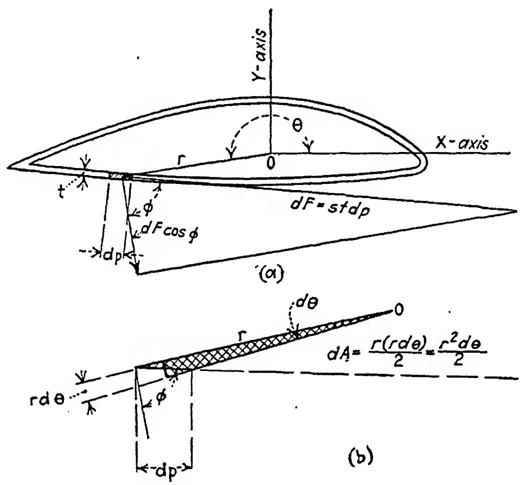


FIG. 90.—Stresses in the walls of a tube.

The relationship between E_s, the modulus of elasticity, E, and Poisson's ratio σ is

$$E_s = \frac{E}{2(1 + \sigma)} \tag{425}$$

In general σ for aircraft materials is approximately 0.25. Hence E_s may be assumed about $\frac{2}{5}$ E.

185. Torsional Stress in a Thin-walled Tube.—Referring to Fig. 90, we have a tubular section with thickness of wall t. We let F represent the total shear force per unit length at any point. This total shear force also exists perpendicular to F as represented in the figure, that is, parallel to the Z-axis of the airfoil. F must be constant around the contour of the airfoil to satisfy the requirements of equilibrium; thus

$$F = ts = \text{constant} \tag{426}$$

in which t is the thickness and s is the unit shear stress.

Taking the summation of moments about an assumed axis O, we have the torque,

$$dM = r(dF) \cos \phi = (rst \cos \phi)dp \tag{427}$$

We note from Fig. 90*b* that

$$dp = \frac{r(d\theta)}{\cos \phi} \tag{428}$$

so that

$$dM = r^2 st(d\theta) \tag{429}$$

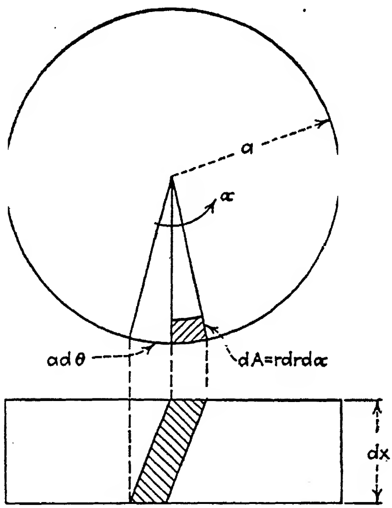


FIG. 91.—Shear in a solid cylinder.

We also note from this figure that

$$r^2(d\theta) = 2(dA) \tag{430}$$

so that, since st is constant,

$$M = 2st\int dA = 2stA \tag{431}$$

in which A is the mean area bounded by the inner and outer curves.

186. Torsional Stress in a Solid Round Rod.—With reference to Fig. 91, the torsional moment M is expressed thus:

$$M = \int srdA \tag{432}$$

Since

$$s:r = s_{(\max.)}:a \tag{433}$$

therefore

$$M = \frac{s_{(\text{max.})}}{a}\int r^2 dA \qquad (434)$$

Let $\int r^2 dA = J$, the polar moment of inertia of the area, then

$$s_{(\text{max.})} = \frac{Ma}{J} \qquad (435)$$

The units are inches and pounds.

187. Angle of Twist of a Solid Round Rod.—With reference to Fig. 91, by definition,

$$E_s = \frac{s_{(\text{max.})}}{a(d\theta)/dx} = \frac{s_{(\text{max.})}}{a}\frac{dx}{d\theta}. \qquad (436)$$

from which

$$= \frac{s_{(\text{max.})}}{aE_s}\int_0^L dx = \frac{s_{(\text{max.})}}{aE_s}L \qquad (437)$$

Substituting (435) in (437)

$$\theta = \frac{M}{E_s J}L \qquad (438)$$

The units are radians, inches, and pounds.

188. Other Compact Solid Rods.—Except for a few simple cross sections of torsion members, such as the circle, the ellipse, the square, etc., precise torsion formulas have not been derived. The mathematics involved in the theory of elasticity in torsion becomes too complicated for the scope of this text. The student is referred to texts on the mathematical theory of elasticity.

For an elliptical cross section we have

$$\theta = \frac{4\pi^2 JM}{E_s A^4}L \qquad (439)$$

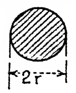
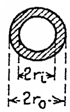
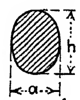
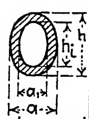
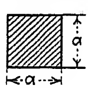
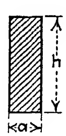
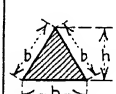
in which A is the area of the cross section in square inches.

For other compact sections, approximately,

$$\theta = \frac{40JM}{E_s A^4}L \qquad (440)$$

189. Torsional Stress in Various Cross Sections.—The student should note especially that the stress formulas in the preceding paragraph and in the following table give the stress which is

TABLE XVIII.—EQUATIONS OF TORSION[1]

No.	Cross section	Moment (torsion)	Shear stress
1		$M = \frac{\pi}{2} r^4 E_s\theta$ $M = \frac{\pi}{2} r^3 s$ $= \frac{Js}{r}$	$s_{(\text{max.})} = rE_s\theta$ $= \frac{Mr}{J}$
2		$M = \frac{\pi}{2}(r_0^4 - r_i^4)E_s\theta$	$s_{(\text{max.})} = r_0 E_s\theta$ $= \frac{Mr_0}{J}$
3	h:a = n:1 n>1	$M = \frac{\pi}{16}\left(\frac{n^3}{n^2+1}\right)a^4 E_s\theta$ $= \frac{\pi}{16} n a^3 s_{(\text{max.})}$ $M^* = \frac{1}{4\pi^2}\frac{E_s\theta A^4}{J}$, A = area	At end of small axis: $s_{(\text{max.})} = \frac{n^2}{n^2+1} aE_s\theta$ $= \frac{16M}{\pi n a^3}$ At end of large axis: $s = \frac{1}{n} s_{(\text{max.})}$
4	h:a = h_i:a = n:1 n>1	$M = \frac{\pi}{16}\left[\frac{n^3}{n^2+1}\right](a^4 - a_i^4)E_s\theta$ $= \frac{\pi}{16} n \frac{(a^4 - a_i^4)}{a} s_{(\text{max.})}$	At end of small axis: $s_{(\text{max.})} = \left[\frac{n^2}{n^2+1}\right] aE_s\theta$ $= \frac{16Ma}{\pi n(a^4 - a_i^4)}$ At end of large axis: $s = \frac{1}{n} s_{(\text{max.})}$
5		$M = 0.1404a^4 E_s\theta$ $= 0.208a^3 s_{(\text{max.})}$ $M^* = 0.0234\frac{E_s\theta A^4}{J}$	At the middle of the side: $s_{(\text{max.})} = 0.6753aE_s\theta$ $= \frac{M}{0.208a^3}$ At corners, $s = 0$
6	h:a = n:1 4>n>1	$M = n\psi a^4 E_s\theta$ in which $n\psi = \frac{1}{3}\left(n - 0.630 + \frac{0.052}{n^4}\right)$ For a section between a square and a flat sheet $M^* = K\frac{E_s\theta A^4}{J}$ K varies from 0.0234 for square to 0.0278 for flat sheet.	At the middle of the long side: $s_{(\text{max.})} = \psi_1 aE_s\theta$ in which $\psi_1 = 1 - \frac{0.65}{1+n^3}$ At corners, $s = 0$
7		$M = \frac{h^4}{15\sqrt{3}} E_s\theta$ $= \frac{b^4}{46.188} E_s\theta$ $M^* = 0.0222\frac{E_s\theta A^4}{J}$	In the middle of the side: $s_{(\text{max.})} = \frac{h}{2}E_s\theta = \frac{b}{2.309}E_s\theta$ At corners, $s = 0$
8	For any compact section without re-entrant-angles	Approximately $M^* = K\frac{E_s\theta A^4}{J}$	$s = \frac{Mv}{J}$ where v is the distance from the axis of twist to the fiber. At corners and sharp edges, $s = 0$.

* See "Applied Elasticity" by Prescott, J., page 177, 1924, Longman, Green and Company.

E_s = modulus of elasticity in shear. s = shear stress, lb. per square inch.
θ = angle of twist per unit length, radians. J = polar moment of inertia.
M = torsional moment, in.-lb. A = area of cross section, sq. in.

[1] Die Lehre der Drehungsfestigkeit von Dipl.-Ing. Constantin Weber, Forschungsarbeiten auf den gebiete Ingenieurwesens, 1921.

developed *below the elastic limit of the material* or *below the buckling strength of the thin-walled tubes.* They do not give the allowable strength—that is, the breaking strength.

190. Torsional Rigidity of a Shell-type Monoplane Wing.—In the designing of cantilever monoplane wings to preclude torsional wing flutter, as has been demonstrated experimentally, torsional rigidity is an important factor. The torsional rigidity of a wing is also important with respect to the maintenance of a constant relative angle of attack throughout the span of the wing; we must also know its magnitude in order to compute the torsional period of the wing.

In general, the contribution of the spars, in a shell type of wing, to the torsional rigidity is of secondary importance; hence we omit their effects from our calculations. The most uncertain quantities in the calculations are the modulus of elasticity in shear of the stressed-cover material and the degree of buckling in the cover. However, a few experiments on wings of the type under discussion should afford a basis for determining coefficients which will account for discrepancies between theory and practice.

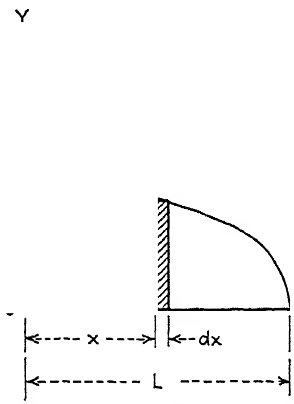


FIG. 92.—Integration of torsion curve.

191. Angle of Twist of a Shell Wing.—We assume that the entire rigidity in torsion of the shell wing is due to the shell, and that the bending stresses are negligible.

In the development of the formula for the angle of twist in terms of the applied torque and the physical characteristics of the material, we resort to the method of energy. With reference to Fig. 92, the applied torque varies along the semi-span L of the wing, and is thus a function of x. The total torque at x is M, and

$$M = \int_x^L q dx \tag{441}$$

The integral can be readily evaluated by approximate methods. We shall represent the angle of twist at x by θ. θ is thus a function of x. Now the external energy *per unit length* expended in

the twisting of the wing is $\frac{q}{2}\theta$, and the energy expended per length dx is

$$du = \frac{q}{2}\theta dx \tag{442}$$

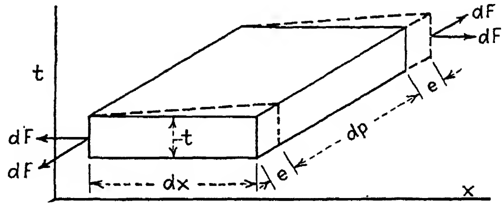


FIG. 93.—Shear in elementary surface area of wing skin.

from which the total energy expended from x to the end of the wing is

$$u = \int_x^L \frac{q}{2}\theta dx \tag{443}$$

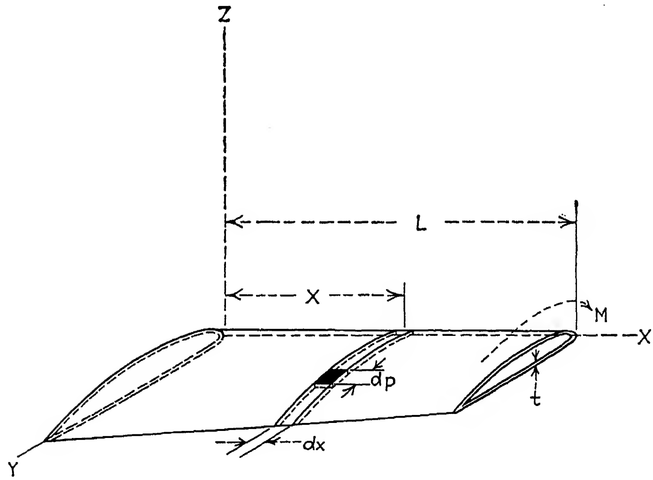


FIG. 94.—Elements of a shell wing.

If M is a concentrated torque applied at the end of the wing and θ_0 is the angle of twist at the end in radians, the total expended energy in the twisting of the wing is

$$u = \frac{M}{2}\theta_0 \tag{444}$$

192. Elastic Potential Energy.—We shall now set up an equation of the potential energy stored in the shell of the wing in the twisted position; this we shall equate to equation (443) and solve for θ

In Fig. 93 is sketched a differential block taken from the shell of the wing as shown in Fig. 94. To this block, a shear force dF is applied. The magnitude of dF is stated thus:

$$dF = st(dp) \tag{445}$$

in which s is the unit shear stress, t is the thickness of the shell, and $t(dp)$ is the area on which dF acts. The energy stored in the differential block because of the motion of dF through a distance e is

$$d(du) = \frac{dF}{2}e \tag{446}$$

We note that the shear modulus of elasticity E_s is expressed by

$$E_s = \frac{s}{(e/dx)} = \frac{sdx}{e} \tag{447}$$

from which

$$e = \frac{sdx}{E_s} \tag{448}$$

Substituting (448) and (445) in (446), we obtain

$$d(du) = \frac{s^2t(dx)(dp)}{2E_s} \tag{449}$$

from which

$$u = \int_x^L \int_0^c \frac{s^2t}{2E_s}dxdp \tag{450}$$

in which the limit c is the periphery of the wing section at x. The thickness t and the shear stress s may be a function of both x and s. Equating (450) and (443), we obtain

$$\frac{1}{2}\int_x^L q\theta dx = \frac{1}{2E_s}\int_x^L \int_0^c s^2tdxdp \tag{451}$$

From equation (431) we have

$$st = \frac{M}{2A} \tag{452}$$

in which A is the area enclosed by the shell at section x. Substituting (452) in (451), we obtain

$$\frac{1}{2}\int_x^L q\theta dx = \frac{1}{2E_s}\int_x^L\int_0^c \frac{M^2}{4A^2t}dxdp \tag{453}$$

Substituting the value of M from equation (441) in equation (453), we obtain

$$\int_x^L q\theta dx = \frac{1}{4E_s}\int_x^L\int_0^c \frac{\left[\int_x^L qdx\right]^2}{A^2t}dxdp \tag{454}$$

193. Approximate Solution of the Energy Equations.—Equation (454) appears to be formidable, but an approximate solution can be readily obtained. If we take ΔL for L and make x zero, we can assume the variables to be constant for this delta-length and equal to the average value. We can also assume that t is constant around the periphery of the wing, since this will undoubtedly be the case in the practical type of construction. Equation (454) thus becomes

$$q\Delta\theta\int_0^{\Delta L} dx = \frac{\left[q\int_0^{\Delta L}dx\right]^2}{4A^2tE_s}\int_0^{\Delta L}\int_0^c \tag{455}$$

Integrating, we obtain

$$q(\Delta\theta)(\Delta L) \qquad \frac{q^2(\Delta L)^2(\Delta L)c}{4A^2tE_s} \tag{456}$$

from which

$$\Delta\theta = \frac{q(\Delta L)(\Delta L)c}{4A^2tE_s} \tag{457}$$

The application of equation (457) is as follows: Suppose, for instance, the semi-span of a cantilever stressed-skin tapered wing is 20 ft. We shall take ΔL as 1 ft. (12 in.) and compute $\Delta\theta$ for each increment. The total angle of twist at the tip is

$$\Delta\theta_0 = \Delta\theta_1 + \Delta\theta_2 + \Delta\theta_3 + \qquad + \Delta\theta_{20} \tag{458}$$

The quantities for the determination of $\Delta\theta_1$ will be computed from a section of the wing 6 in. from the tip, for $\Delta\theta_2$ from a section 18 in. from the tip . . . and for $\Delta\theta_{20}$ from a section 6 in. from the root section. Note that $Q_1 = q_1\Delta L$, $Q_2 = Q_1 + q_2\Delta L$, etc.

194. Torsion in Box Beams of Rectangular Cross Sections.—In the case of a box beam of rectangular cross section with sides of thickness t_1 and height b and top and bottom of thickness t_2 and width d, equation (454) may be written,

$$\int_x^L q\theta dx = \frac{1}{4E_s}\int_x^L\left[2\int_0^b\frac{\left[\int_x^L qdx\right]^2}{A^2t_1}dp + 2\int_0^d\frac{\left[\int_0^L qdx\right]^2}{A^2t_2}dp\right]dx \tag{459}$$

From equation (459) we determine an equation corresponding to equation (457) as follows:

$$\Delta\theta = \frac{\left[\sum_x^L(q\Delta L)\right](\Delta L)b}{2A^2t_1E_s} + \frac{\left[\sum_x^L(q\Delta L)\right](\Delta L)d}{2A^2t_2E_s} \tag{460}$$

For the torsion angle produced at the end of a uniform beam by a torque M applied at the end of the beam, we write M for $\sum_x^L q\Delta L$ and L for ΔL in equation (460). Thus

$$\theta = \frac{MLb}{2A^2E_st_1} + \frac{MLd}{2A^2E_st_2} = \frac{M}{2A^2E_s}\left[\frac{b}{t_1} + \frac{d}{t_2}\right]L \tag{461}$$

We note that $A = bd$.

An equation similar to (461) and derived from (457) is

$$\theta = \frac{Mc}{4A^2tE_s}L \tag{462}$$

195. Angle of Twist of a Round Tube.—The calculation of the torsion angle of a round tube is a special case of the above process. We note that for a round tube

$$\theta = \frac{M}{E_sJ}L \tag{463}$$

Let us see if equation (463) is a special case of equation (462) and thus check our theory for a round tube:

$$J = \frac{\pi r_2^4}{2} - \frac{\pi r_1^4}{2} = \frac{\pi}{2}(r_2^4 - r_1^4) \tag{464}$$

in which r_1 and r_2 are the inner and outer radii, respectively. Expanding equation (464),

$$J = \frac{\pi}{2}(r_2^2 + r_1^2)(r_2 + r_1)(r_2 - r_1) \tag{465}$$

Now $\frac{\pi}{2}(r_2^2 + r_1^2)$ is the average inclosed area A, $\frac{r_2 + r_1}{2}$ is the average radius r, and $r_2 - r_1$ is the thickness t. Thus

$$J = 2Art \tag{466}$$

and

$$\theta = \frac{M}{2AE_srt}L = \frac{2\pi rM}{4\pi r^2AtE_s}L = \frac{Mc}{4A^2tE_s}l \tag{467}$$

which checks equation (462).

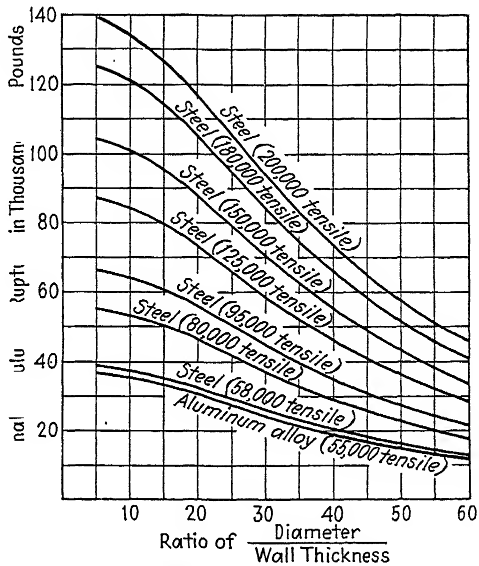


FIG. 95.—Shearing strength of steel and aluminum-alloy tubes.

196. Torsional Strength of a Round Tube of Thin Walls.—Tubes of relatively thick walls fail through shear. Tubes of relatively thin walls fail by buckling of the walls—that is, by elastic instability. The failure stress, as has been previously noted, is a function of the ratio of diameter D to thickness of wall t. The curve of failing stress s as a function of the ratio D/t may be represented by a curve similar to the combined Johnson's parabolic and Euler's curves for a strut. It is pointed out in *Air Corps Information Circular* No. 641, "The Allowable Stress in Tubes Subjected to Torsion," a printing of A.D.M. 1042 prepared by the author, that for values of D/t less than 60, the following empirical formula would fit the data fairly well and give a conservative value of the allowable stress s_a:

$$s_a = \frac{1600 s_0}{\left(\frac{D}{t} - 2\right)^2 + 1600} \tag{468}$$

in which s_0 is the unit-shearing strength of the solid shaft.

A check shows that the curves of Fig. 95 specified by the Aeronautics Branch of the Department of Commerce in their *Aeronautics Bulletin 7-A* may be duplicated by equation (468). Further experiments carried out at the University of California for values of D/t from 50 to 1,100 show that, while this formula is satisfactory, but rather conservative, for ratios of D/t less than

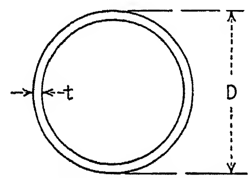


Fig. 96.—Dimensions of tube.

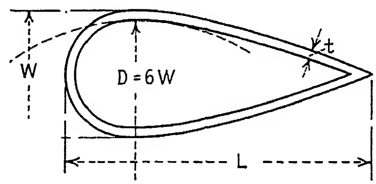


Fig. 97.—Streamlined strut.

60, it is entirely too conservative for larger values of D/t. However the formula

$$\frac{40 s_0}{\left(\frac{D}{t} - 2\right) + 40} \tag{469}$$

recommended in the above-mentioned report, represents the data with fair conservativeness and accuracy for ratios of D/t up to 1,100 and probably for even larger values.

197. Torsional Strength of Various Tubes.—In the absence of experimental data, we may approximate the torsional strength of tubes of streamlined, elliptical, and other sections by the following method: The tubes will probably buckle at the section of the circumference which has the greatest radius of curvature. Using this radius, we may compute a value of D/t as $2r/t$ (see Fig. 97). With this ratio known, s_a may be calculated from equation (468) or (469). The stress developed may be computed by equation (431).

198. Combined Axial Compression and Torsion in a Thin-walled Tube.—At the present writing, very few data are available on this problem. Since a theoretical analysis cannot be

relied upon, in this case, for design purposes, it seems practical to attempt to obtain an approximation of the allowable combined stress based upon the allowable stress in shear and the allowable stress in axial compression or in pure bending.

It appears that, since there is a variation of 50 to 100 per cent in data obtained in the simple torsion, compression, and bending experiments, we are not justified in presenting elaborate formulas or methods for design purposes. It appears that we may obtain reasonable and conservative design values by the following method:

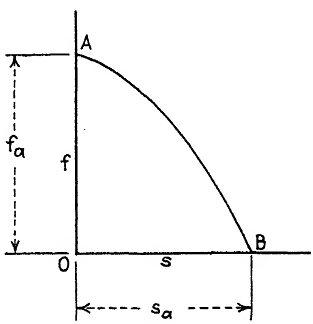


FIG. 98.

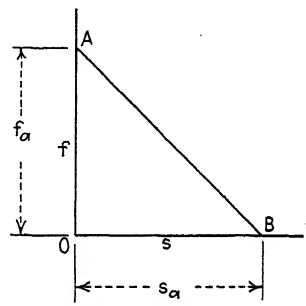


FIG. 99.

In this analysis, let

s_a = allowable shear stress, when shear alone is assumed acting.

f_a = allowable axial compressive stress.

f_c = combined allowable shear stress and axial compressive stress.

s = shear stress [see equation (431)].

f = compressive stress.

[See equations (468), (469), and (489) for estimated values of s_a. See equations (476) and (482) for estimated values of f_a. Equation (476) applies to pure axial compression, and equation (482) to axial compression due to bending.]

If, as in Figs. 98 and 99, we let f be the ordinate of the graph and s the abscissa, the allowable combined stress would probably be represented with fair accuracy by a curve similar to AB, for example, by the ellipse

$$\frac{s^2}{s_a^2} + \frac{f^2}{f_a^2} = 1$$

or by the cosine curve

$$f = f_a \cos \frac{\pi s}{2 s_a}$$

The straight line AB of Fig. 99, however, is the most conservative of the three. In this curve

$$f = \frac{f_a(s_a - s)}{s_a}$$

In attempting to check these equations with experimental data, the average curve of the data should give about twice the value of the formulas, since f_a and s_a are selected to represent, conservatively, the minimum values from experiments varying in results about 100 per cent for the same ratios of D/t.

The above formulas do not give the total combined stress f_c. We may obtain a design value for this by reference to Fig. 100. In this graph, the abscissa is taken as the ratio between the shear stress and the combined shear and axial stress

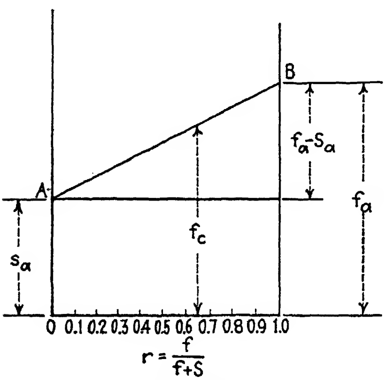


FIG. 100.

$$r = \frac{f}{f + s}$$

When $f = 0$, $r = 0$; thus this ordinate represents the allowable shear stress. When $s = 0$, $r = 1$; thus this ordinate represents the allowable axial stress. Now f_c will lie on a curve drawn from A to B. The nature of this curve we do not know. It is reasonable to assume a straight line. In this case

$$f_c = s_a + (f_a - s_a)r \tag{470}$$

In combined bending and torsion, buckling will occur as shown in Fig. 101, that is, at an angle θ to the axis of the tube as shown in Fig. 102.

Selected References

1. PRESCOTT, JOHN: "Applied Elasticity," Longmans, Green and Company, London, 1924.

2. TIMOSHENKO, S.: "Theory of Elasticity," McGraw-Hill Book Company, Inc., New York, 1934.

3. TIMOSHENKO, S., and J. M. LESSELLS: "Applied Elasticity," Westinghouse Technical Night School Press, East Pittsburgh, Pennsylvania, 1925.

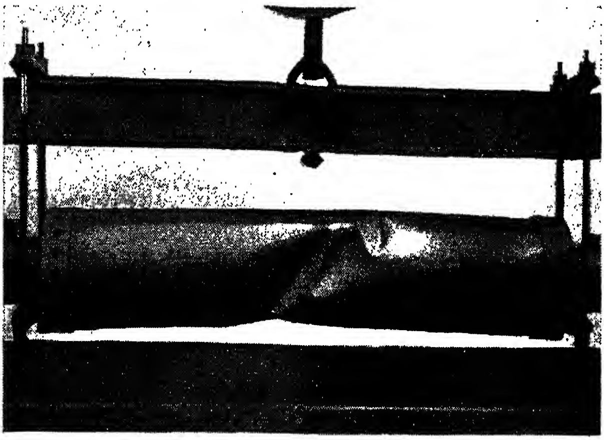


FIG. 101.—Failure of thin-walled tube under combined torsional and bending load.

4. YOUNGER, J. E.: Miscellaneous Collected Airplane Structural Design Data, Formulas, and Methods, *Air Corps Information Circular* 644, 1930.

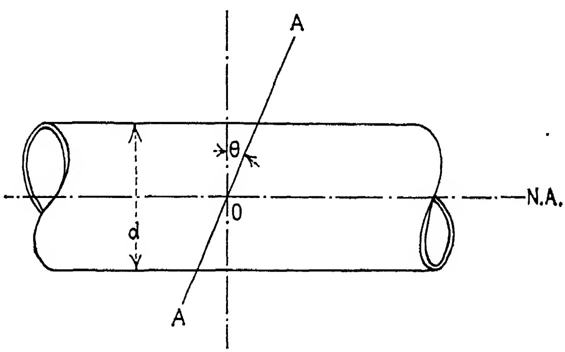


FIG. 102.—Torsion and bending in tube.

5. YOUNGER, J. E., and B. M. WOODS: "Dynamics of Airplanes and Airplane Structures," John Wiley and Sons, Inc., New York, 1931.
6. YOUNGER, J. E.: The allowable stress in tubes subjected to torsion, *Air Corps Information Circular* 641, Oct. 1, 1929.

SECTION IV

SPECIAL PROBLEMS IN THE DESIGN OF METAL AIRPLANES

CHAPTER XIII

PRINCIPLES OF DESIGN OF SHEET-METAL CONSTRUCTION

199. Fundamental Concepts.—The study of thin sheet-metal structures is a study of the strength of tubes with thin walls, in tension, compression, shear, and bending. We note that the strength of a tube in *compression* (as a column) and in *bending* is proportional to the moment of inertia of the cross-sectional area about a transverse neutral axis. The strength in torsion is proportional to the polar moment of inertia of the cross-sectional area about the longitudinal axis. Both these moments of inertia, for the same weight per unit length of tube, are increased by increasing the diameter D for constant cross-sectional area. For example, with reference to Figs. 103*a* and 103*b*, make $r_1 = 2r_0$; then t is obtained for the constant area A from the equation

$$A = 2\pi r_0 t_0 = 2\pi r_1 t_1 \tag{471}$$

Thus

$$t_1 = \frac{r_0}{r_1} t_0 = \frac{r_0 t_0}{2r_0} = \frac{t_0}{2}$$

Since

$$2\pi r_0^3 t_0 \qquad \text{and} \qquad J_1 = 2\pi r_1^3 t_1$$

then

$$\frac{\frac{1}{2}J_1}{\frac{1}{2}J_0} = \frac{2\pi r_1^3 t_1}{2\pi r_0^3 t_0} = \frac{(2r_0)^3(t_0/2)}{r_0^3 t_0} = 4$$

The second tube is therefore four times as strong as the first tube, whether considered as a column, as a torque tube, or as a beam, for the same fiber stress.

200. Types of Thin Sheet-metal Structures.—If the tube of Fig. 103 be increased to such size and made such shape that the resulting shell is an airplane fuselage or an airplane wing, the structure is referred to as *monocoque*, *stressed-skin*, or *shell* type of construction. However, strictly as defined, no such airplane structure has ever existed. Since the thin sheets of such large

size would be quite unstable, it is necessary that supporting structures be added in the form of bulkheads or diaphragms, and compression ribs or stringers (see Fig. 104). This type of structure is called *semi-monocoque* or *semi-stressed* skin.

The application of the principles of thin sheet-metal design to the construction of beams and struts of corrugated strips of stainless steel is generally referred to as *corrugated-strip-steel construction* (see Fig. 105). The same principle of course is also applicable to duralumin and other metals.

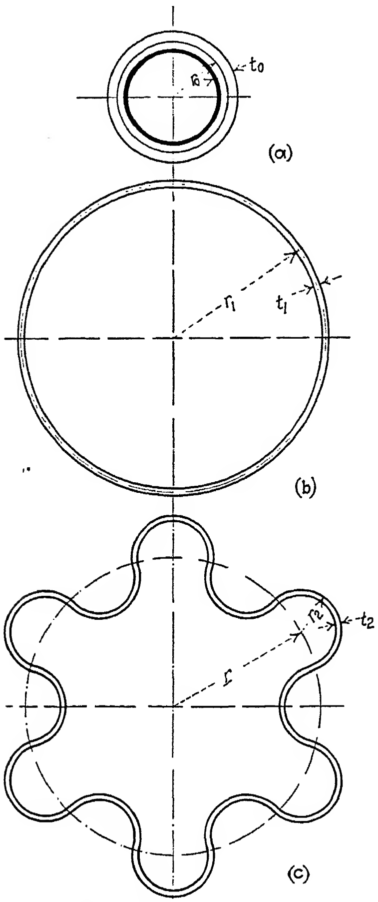


FIG. 103.—Tubes of equal cross-sectional area as structural members.

201. Compressive Strength of Thin Curved Sheets.—If a thin-walled tube is subjected to an axial load, it may fail in one or both of two ways: (1) by buckling as a column, or (2) by local buckling or wrinkling of the thin walls. An example of wrinkling is shown in Fig. 106.

If we assume that the fiber stress which will cause wrinkling is a function of the modulus of elasticity E, the radius of curvature R, and the thickness t, we write

$$f = f(E, R, t) \qquad (472)$$

Applying the principle of dimensions, we write

$$f = KE^aR^bt^c \qquad (473)$$

The absolute dimensions of these quantities are: of f, a force per unit area, M/T^2L; of E, a force per unit area, M/T^2L; of R, a length L; and of t, a length L. We therefore write equation (473) as

$$f = KE^aR^bt^c \qquad \frac{M}{T^2L} \qquad \frac{M^a}{T^{2a}L^a}L^bL^c \qquad (474)$$

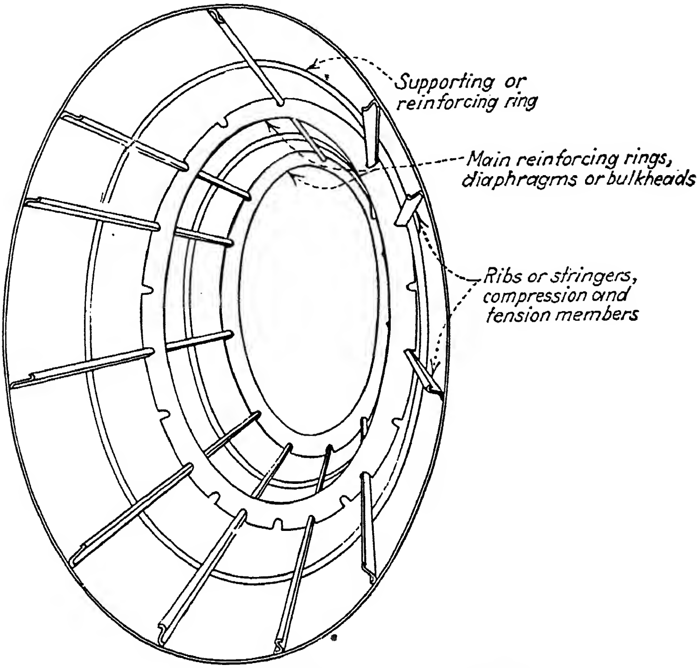


FIG. 104.—Semi-monocoque fuselage.

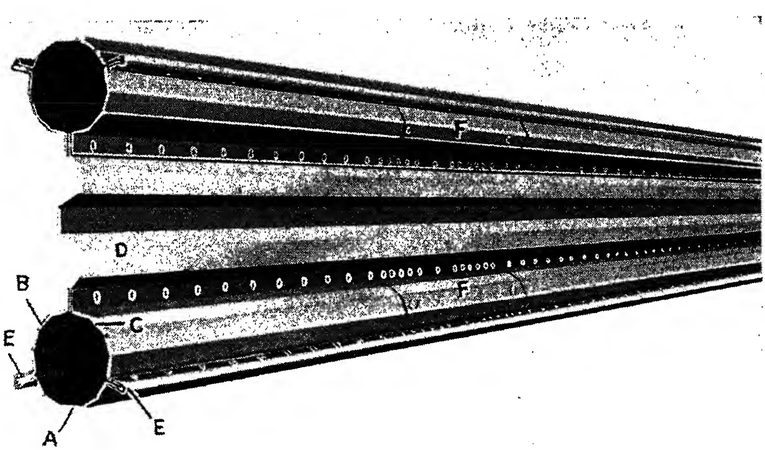


FIG. 105.—Showing a type of single web spar, developed and extensively used by Armstrong Whitworth Aircraft, Ltd.

Equating exponents, we have for M,

$$1 = a, \qquad a = 1$$

for T, $-2 = -2a$, $a = 1$
and for L, $-1 = -a + b + c$
or $b + c = 0$, $b = -c$
Thus equation (473) may be written

$$f = KE\left(\frac{t}{R}\right)^c \tag{475}$$

Experimental data and a mathematical analysis of the problem show that c is approximately 1. When f/E is plotted as a function of R/t for available experimental data, K is found to vary, in the same laboratory, 100 per cent or more for the same value of R/t. This is due to the difficulty of obtaining ideal test conditions. It is reasonable to suppose, therefore, that worse variations will occur in practical structures. What appears to be a conservative design value is $K = 0.12$. $K = 0.24$ represents about the average data. We write as our conservative design formula for axial compression (see References 2, 4, and 9):

$$f = 0.12E\frac{t}{R} \tag{476}$$

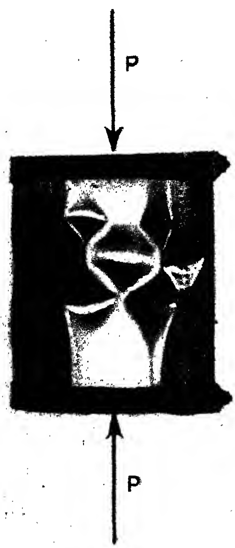


FIG. 106.—Wrinkling of thin curved sheets.

202. Equal Strength in Buckling and Wrinkling.—If we consider an Euler strut of thin walls, we have

$$\frac{P}{A} \quad \frac{c\pi^2 E}{(L/\rho)} = f = 0.12E\frac{t}{R} \tag{477}$$

in which L is the length of the strut and ρ is the radius of gyration. For a pin-ended strut $c = 1$. Solving for L/ρ in terms of t/R, we have, approximately,

$$\frac{L}{\rho} = 9\sqrt{\frac{R}{t}} \tag{478}$$

For example, if a tube of 4 in. diameter has a wall thickness of 0.020 in., L/ρ at which wrinkling occurs is 90. For lengths of tube less than this, failure will be by wrinkling, and for lengths greater, by buckling.

Equation (478) is confirmed by the chart for *buckling* and *wrinkling* strength of corrugated sheets (Fig. 112). For the curve $c = 1$, wrinkling occurs at $L/\rho = 90+$ for a value of $R/t = 100$, and at $L/\rho = 45-$ for a value of $R/t = 25$.

For $c = 2$,

(479)

For $c = 3$,

$$\frac{L}{\rho} = 9\sqrt{3\frac{R}{t}} \tag{480}$$

203. Use of Corrugations.—Equation (476) tells us that, for a constant thickness t, f is increased by decreasing R. We may

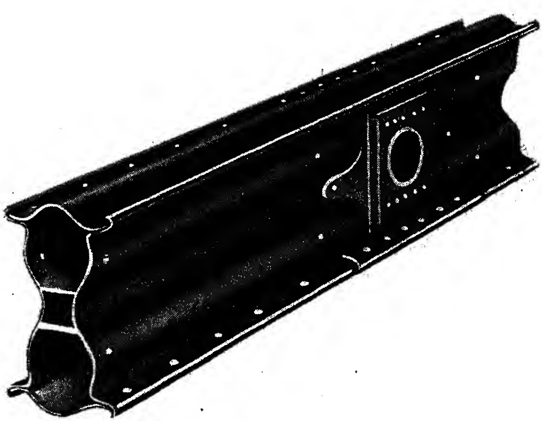


FIG. 107.—Steel box spar produced by Armstrong Whitworth Aircraft Ltd., 1918. (*Courtesy of A. T. S. Ltd.*)

therefore, with reference to Fig. 103*c*, improve the strength-weight ratio of a tube by longitudinal corrugations. The design of such a wing beam as shown in Fig. 107 is based on this principle, as is also the box beam shown in Fig. 108.

204. Strength of Corrugations.—*Aeronautics Bulletin* 7-*A* gives the following summary of calculations for the strength of corrugated stress members:

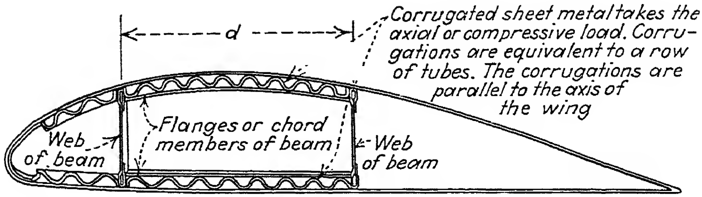
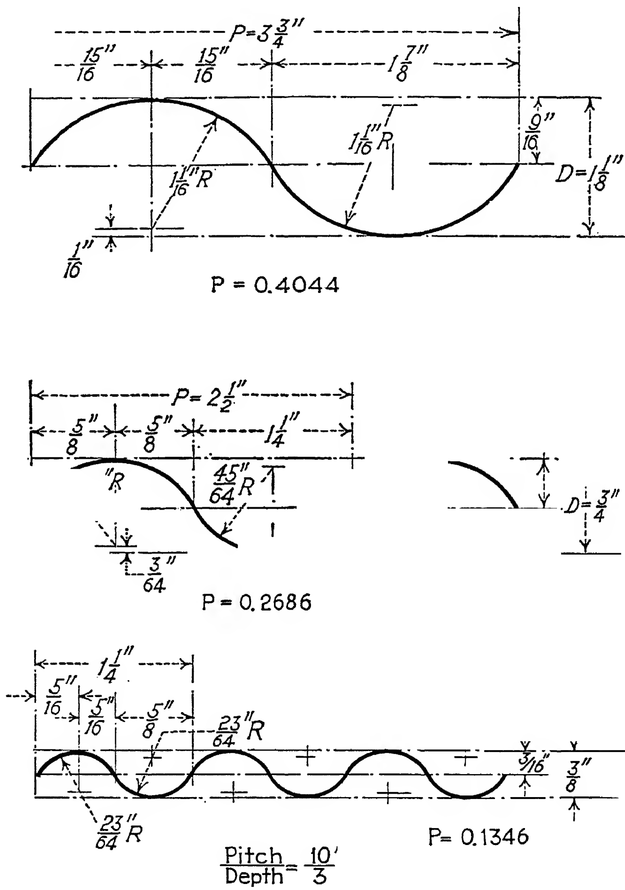


FIG. 108.—Modern stressed-skin wing.



$$\text{Radius of gyration} = 0.719 \times \frac{D}{2}$$

$$\text{Curvilinear length of one corrugation} = 1.228 \times \text{pitch}$$

$$\frac{R}{P} = 0.282$$

FIG. 109.—Standard corrugations in aluminum-alloy sheet.

Corrugated Aluminum-alloy Sheet:

1. *Shape.*—The dimensions of standard corrugated aluminum-alloy sheet are shown in Fig. 109.

2. *Properties.*—The geometrical properties of the standard corrugations are:

$$I = 0.158tD^2$$
$$\rho = 0.359\ D$$
$$W_e = 1.228\ W$$
$$R = 0.282\ P$$

where I = moment of inertia in.[4] per inch width:
ρ = radius of gyration.
D = depth of corrugation.
P = pitch of corrugation.
W = width of corrugated sheet.
W_e = width of equivalent flat sheet.
R = radius of curvature of corrugation.
t = thickness of sheet.

The section properties of other corrugations, in terms of their pitch to depth ratio, are shown in Fig. 110.

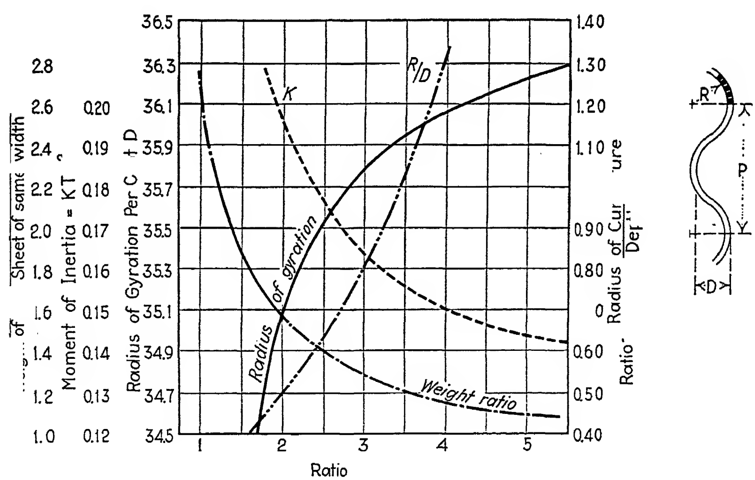


FIG. 110.—Section properties of corrugated sheet.

3. *Column strength.*—Heat-treated corrugated aluminum-alloy sheet in column action may be designed by the Euler-straight line column formula for duralumin, provided that the unit stress as determined by the column formula does not exceed the buckling stress for the corrugated sheet.

Figure 111 gives the buckling (wrinkling) stress as a function of R/t, the ratio of the radius of the curvature of the corrugation to the thickness of the sheet. It should be kept in mind that the buckling (wrinkling) stress is independent of L/ρ and of the restraint coefficient c.

Figure 112 expresses the properties of corrugated aluminum-alloy sheet as a function of both L/ρ and R/t. Its use can best be made clear by an example.

Consider a sheet having an R/t of 50 and $c = 1$.

If L/ρ is 80, the allowable P/A is 15,000 lb. per square inch.

If L/ρ is 40, the allowable P/A is 20,000 lb. per square inch.

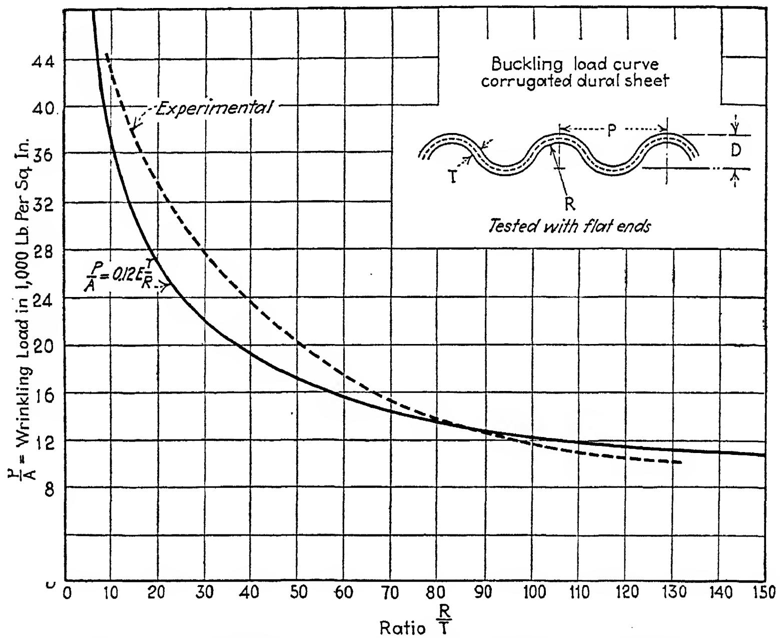


FIG. 111.—Buckling stresses in corrugated aluminum-alloy sheet.

When the construction is such that the corrugated sheet is simply laid over the ribs, or bulkheads, and riveted thereto without additional metal covering, the value of c shall be 1, and L shall be the distance between bulkheads. If the edges of the sheet are supported by members sufficiently strong to prevent local failure, and the width of the sheet is less than three-fourths of the length, c may be increased not to exceed 1.5.

No rules can be laid down governing the fixity coefficients to be employed in the many combinations of length, end condition, intermediate restraint, and attached parts or covering possible in connection

with the structural employment of corrugated sheets. Each case when the stresses appear critical will require separate consideration and possibly the evidence of actual test. However, the following appears to

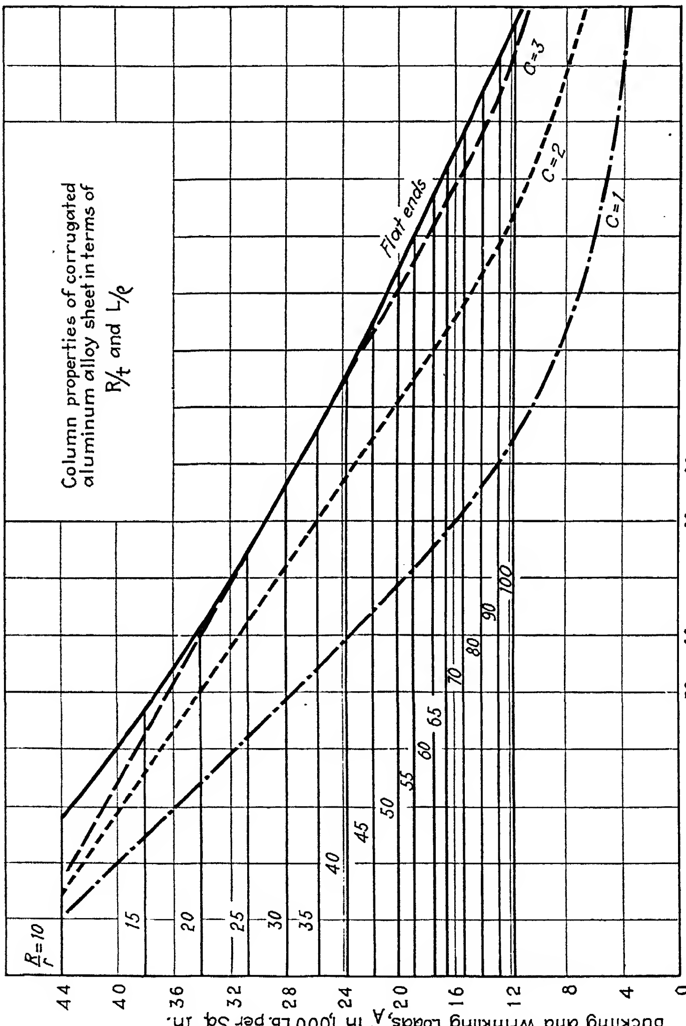


FIG. Column ugat aluminum loy

afford a conservative basis for computing fixity coefficients at the present time.

When the construction is such that an additional flat sheet is laid over the corrugations, as would be the case in wing construction, the value

c shall not exceed 1.5 and L shall be the distance between ribs or bulkheads.

If the construction is such that an additional corrugated sheet forms the covering and this sheet is so attached that it can support the longitudinal corrugations and prevent column action, the value of c may be taken as 3.

In each of the above cases if the buckling stress is the lower stress, it shall be the design stress.

205. Strength of Thin-walled Tubes in Bending.—Since the stresses in general are below the elastic limit of the material, the bending stress (tension or compression) may be computed

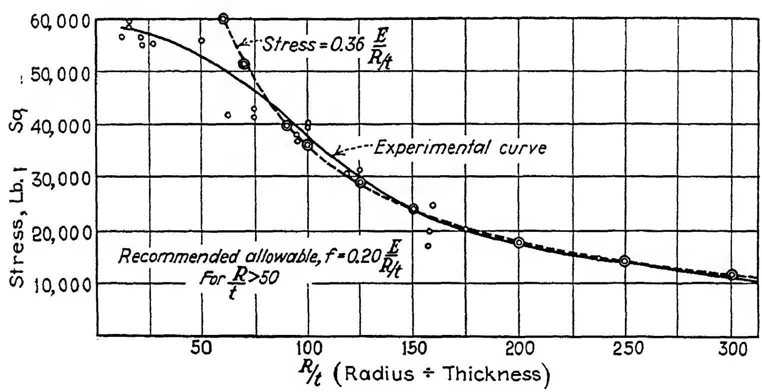


FIG. 113.—Allowable stress in bending: Thin-walled duralumin tubes.

by the modulus-of-rupture formula [equation (203)]. This stress f, however, must be below the allowable stress f_a. The stress f_a is the critical stress which causes wrinkling. In a manner similar to that of Par. 201, it can be shown that

$$f_a = K_b E(t/R)^c \tag{481}$$

in which $c = 1$.

Failure will occur, of course, in compression. Figure 113 shows an experimental curve obtained, under the direction of the author, at the University of California from bending tests on thin-walled cylinders with a maximum radius-to-thickness ratio of about 300. The average constant K_b was found to be approximately 0.36. Experiments performed at Stanford University on large cylinders of radius-thickness ratio of about 2,500 encouraged a recommendation of $K_b = 0.30$. It appears advisable however to use a value for design somewhat comparable to that

used in equation (476). It seems that a value of not over 0.20 would be advisable, that is,

$$f_a = 0.20E\frac{t}{R} \tag{482}$$

This formula is for plain, unsupported tubes. For large tubes using corrugated compression flanges, equation (476) or values from Fig. 111 should be used.

If we let M_a be the allowable bending moment then

$$\frac{M_aR}{I} = 0.20E\frac{t}{R} \tag{483}$$

Since $I = \pi tR^3$, we find that

$$M_a = 0.628Et^2R^2 \tag{484}$$

Thus in all cases

$$M < 0.628Et^2R^2 \tag{485}$$

206. Longitudinal Shear in Bending.—The maximum longitudinal shear occurs in the plane of the neutral axis (see Fig. 114).

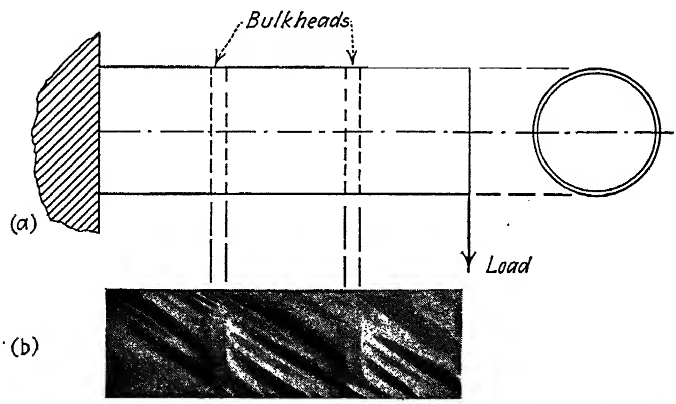


Fig. 114.—Shear in bending of thin-walled tube. (*a*) "Free-body" diagram; (*b*) wrinkling of sides of tube.

Its value may be computed, below the elastic limit of the material, by equation (226). In this case

$$s = \frac{VQ}{Ib} \tag{486}$$

Since V is the vertical shear on the beam, b is $2t$, Q is $2R^2t$, and I is $\pi R^3 t$, we have

$$\frac{2VR^2t}{2\pi R^3t^2} = \frac{V}{\pi Rt} \tag{487}$$

The allowable shear s_a for the critical load may be obtained from equation (468) or (469).
The general form of this equation is

$$s_a = K_s E \frac{t}{R} \tag{488}$$

If s_0 of equation (469) be taken as 30,000 lb. per square inch, E as 10,000,000 lb. per square inch, and the constants in the denominator neglected for large values of R/t, we find K_s to be approximately 0.06. Experiments carried out at the University of California on this type of shear gave an average K_s of 0.07 and a lower limit of approximately 0.06. It appears that this value of K_s will give a reasonable, conservative s_a for practical construction. Thus

$$s_a = 0.06E\frac{t}{R} \tag{489}$$

This equation gives the stress for initial wrinkling of the thin sheet. This is desirable because the resulting distortion of the tube may cause it to fail in compression. The equation applies to tubes with unsupported walls. Unless, however, the walls are supported by a very fine network of supports, K_s should not be taken larger than 0.06 because wrinkling may occur with fully developed lobes in a fairly small area of unsupported surface.

207. Fittings in Monocoque Structures.—The function of a fitting is to transmit stresses from one member of a structure to another member or to other members. The proper design of fittings requires that this transmission of stress be accomplished efficiently. The characteristics of this efficiency we may list as follows:

1. Minimum weight for the accomplishment of the function of the fitting.
2. Transmission of stress without inducing fitting stresses, such as may cause local failures, eccentric loadings, etc.

In the design of a fitting careful consideration should be given to the following features:

1. Ultimate strength of component parts of the fittings with reference to tension, compression, shear, and elastic stability.

2. Method of attachment to members to develop the design strength of the fitting and members, such as riveting, bolting, welding, etc.
3. The rigidity of the fitting.

In framed structures, the rigidity of a fitting receives little consideration since, in general, if a fitting has the ultimate strength necessary, it also has the rigidity necessary. In monocoque structures, however, this condition in general does not exist; a fitting which has only the ultimate strength that is necessary would not in general have the rigidity necessary. As a matter of fact, the problem of rigidity is of such importance in fitting design for monocoque structures that it may be said that *rigidity is the criterion of design.*

208. Distribution of Stress.—In a framed structure we are concerned solely with the transmission of concentrated loads, while in monocoque structures we have the additional requirement that the fitting must transmit stresses received into the fitting as a concentrated load, but discharged as a distributed load. The fitting, in this case, has the double duty of *transmission* and *distribution.* It is this distribution of stress which makes the rigidity requirement necessary.

Let us now consider a few illustrations of these fundamental requirements in design, beginning with the most elementary cases. We note that monocoque structures are in general tubular in form, as, for example, the monocoque fuselage, the stressed-skin wing, the monocoque engine nacelle, control torque tubes, etc. It is apparent, therefore, that a study of transmission of stresses as tension, compression, shear, and bending to a thin-walled tube embodies all the elements of monocoque fitting design. Consider, therefore, the thin-walled tube shown in Fig. 115.

In this figure we have a tube constructed by rolling a thin sheet of metal around three solid bulkheads of great rigidity A, B, and C. When the tube is completed by riveting or welding the longitudinal seam, concentrated loads may be applied as noted in Figs. 116, 117, and 118.

We have thus provided for compression, tension, bending, torsion, and shear in our fittings A, B, and C. In so far as the transmission of stress, concentrated to distributed, is concerned, these fittings are perfect, inasmuch as stresses may be developed in the thin metal equal to or greater than the allowable stresses for an infinite length of the tube. In other words, the fittings

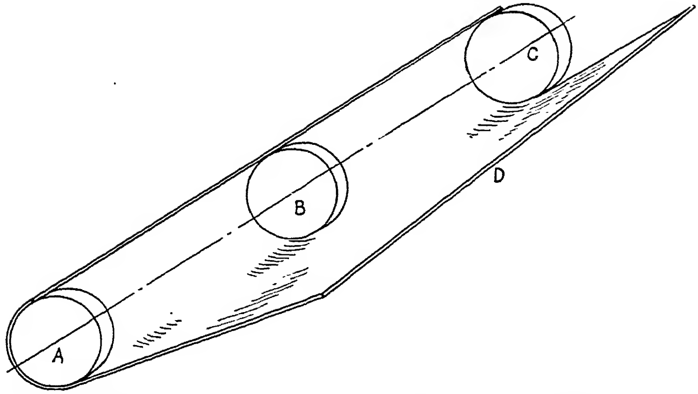


FIG. 115.—Illustrating the fundamental principle of transmission of loads in monocoque structures. The thin sheet *D* is rolled about the bulkheads *A*, *B*, and *C*.

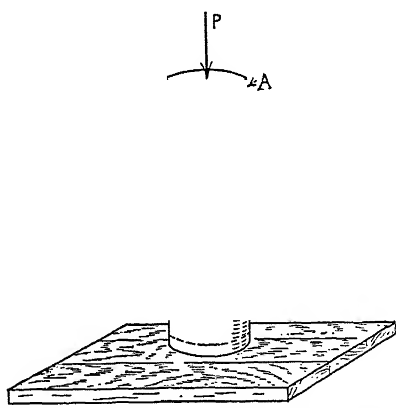


FIG. 116.—Transmission of concentrated axial load to distributed load through rigid bulkhead fitting *A*.

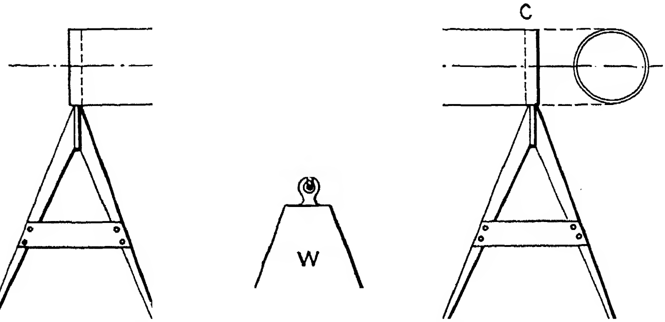


FIG. 117.—Concentrated loads applied to fittings or bulkheads *A*, *B*, and *C*, develops full strength of monocoque tube.

are perfect in so far as they do not introduce *local stresses*, that is, stresses which cause the thin sheet to fail because of improper distribution of the concentrated load.

We note, however, that while the fittings transmit stresses perfectly, they are not in general satisfactory because of excessive weight. Our problem now becomes one of reducing the weight without seriously affecting the transmission and distribution qualities of the fittings. The most efficient fitting will represent a compromise of the problem.

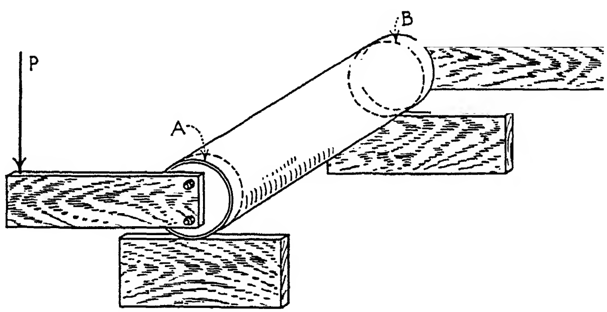


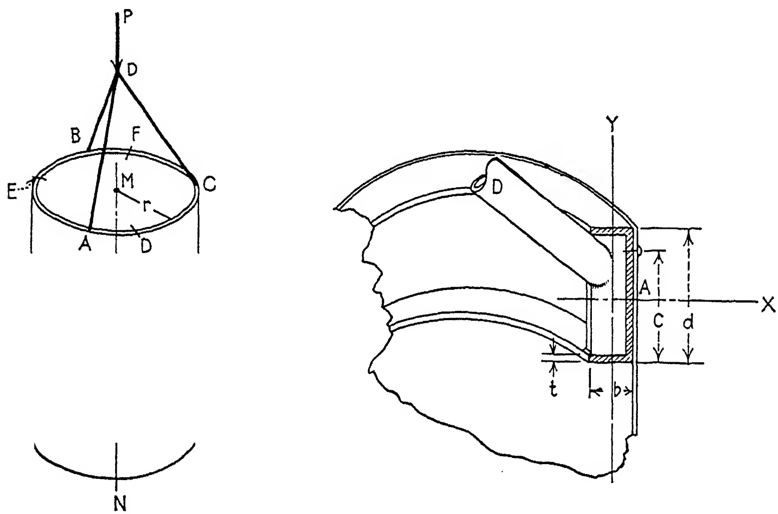
FIG. 118.—Torque and concentrated loads, transmitted to monocoque tube through rigid bulkhead fittings, *A* and *B*.

209. Application in Design.—Before, however, we consider the problem of weight, let us note the application of the bulkhead fitting to airplane structures. *It may be concluded that a rigid bulkhead in a monocoque structure is a perfect fitting for the transmission and distribution of stress.* We accordingly arrive at the following design features in regard to metal monocoque aircraft construction: Wings, tail surfaces, chassis, engine mounts, etc., may be attached to a monocoque fuselage through the medium of rigid bulkheads built into the fuselage for this specific purpose. Likewise we may attach engine nacelles to stressed-skin wings, engine mounts to monocoque nacelles, etc., through rigid bulkheads.

210. Weight Reduction.—We have noted that an efficient fitting represents a compromise between the reduction of weight and the transmission and distribution of stress. Let us now consider the problem of weight reduction, which is the most difficult in design. The question constantly before us is: *How*

light can we make these bulkheads and yet meet the strength and rigidity requirements? A mathematical analysis is difficult, inasmuch as the ultimate strength is involved in conjunction with the elasticity of the material.

211. Axial Compressive Load.—Let us now, to fix ideas, consider a problem of weight saving in design. Let us assume that we are to design a fitting for a simple axial compressive load (see Fig. 119).



(a) (b)

FIG. 119.—Transmission and distribution of concentrated load.

The load P is to be transmitted to the tube MN so that the load per unit of circumference is $KP/2\pi r$. Assume that the allowable stress, that is, the crumpling stress in the walls of the tube, is $0.9P/2\pi r$. The uneven distribution of stress will of course be caused by the nonperfect rigidity of the fitting. The maximum stresses in the thin walls of the cylinder will occur at A, B, and C, and the minimum stresses at points D, E, and F, halfway between the points A, B, and C. A bulkhead in the form of a ring, designed to resist bending about the x-x axis, as shown in Fig. 119*b*, would be desirable. Bending about the y-y axis could be prevented, if necessary, by tie rods A to B, B to C, and C to A. The hoop of Fig. 119*b* must be designed so that the deflection of the circular beam at any point is greater than 90 per cent of the

deflection of the points of application of the loads at A, B, and C. The problem is within the province of the theory of elasticity, since the stresses will, in general, be well within the proportional limit, so that Hooke's law applies.

It is rarely, if ever, that this same condition, in its simple form, occurs in aircraft construction. The thin walls of the tube will undoubtedly be braced by vertical and horizontal supports, so complicating the problem that a model test appears to be the only solution.

An application of this fundamental principle occurs in the design of an engine-mount fitting.

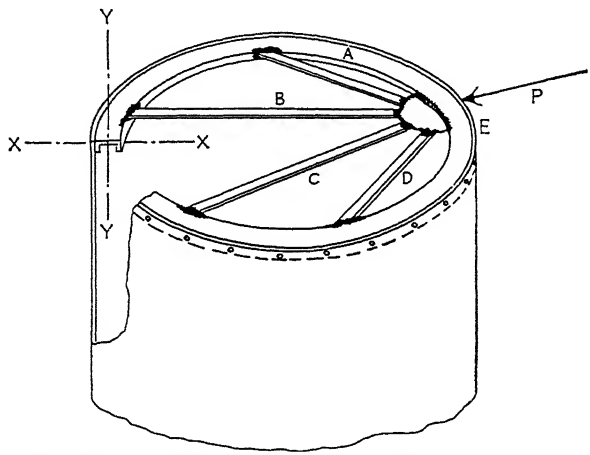


FIG. 120.—Possibility in reduction of weight of a fitting to transmit and distribute a concentrated load.

We may note in passing that the attachment of thin metal to the bulkhead is a problem of importance. If the riveting process is used, the stress may be more evenly distributed by riveting at the top, as shown in Fig. 119*b*, allowing the space designated as *c* as a support for the thin sheet to prevent local wrinkling because of high local rivet stresses.

A concentrated load applied perpendicularly to the axis of the tube, as shown in Fig. 120, may be provided for by a fitting of the general type shown in the figure.

In this case, stiffness in bending about the y-y axis is desirable. This stiffness is improved by supports A, B, C, and D, which at the same time provide against local failure in the fitting at E.

212. Fuselage and Wing Joints.—An interesting and important problem in design is the design of a fitting to connect two similar tubes, the tube to be subjected to bending. Such a fitting may be the connection between the outer panel and the center section of a stressed-skin cantilever wing, or the connection between sections of a monocoque fuselage. It is obviously impossible to acquire a smooth enough fit in the ends of the thin skin to eliminate local failures in compression. Actual contact of the skins, therefore, seems undesirable. Figure 121 shows two commonly used joints: (1) a tension joint, and (2) a shear joint. The rivets and bolts should be designed to carry a load greater than the design loads in the thin sheets A and B, since there is considerable

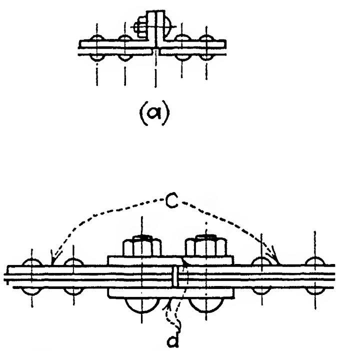


FIG. 121.—Fitting for axial connection of two tubes such as stressed-skin wings. Unless the fitting is very rigid, a bending stress is induced in the thin metal.

uncertainty in the analysis of a fitting. The flanges to be riveted to the thin sheet should be wide enough to afford the proper rigidity to the fitting and to provide ample riveting space. A wide flange would also afford support to the thin sheet against local wrinkling caused by concentrated loads at the rivets.

In the fitting of Fig. 121b the thin sheets A and B are reinforced against local wrinkling by heavier sheets C. These reinforcing sheets may be of about 0.064-in. material for 0.30-in. walled tube with a width of about 3 to 6 in.

High-strength steels such as stainless and nickel steels, if properly protected from corrosive action by bitumastic paint, would be advantageous because of their rigidity (high modulus of elasticity). Figure 122 shows a stressed-skin wing with a fitting similar to the tension type (see Fig. 121a).

213. Fuselage-wing Connection.—The connection of two tubes set at right angles to each other presents an interesting design problem; this is the problem to be solved in the connection of a monocoque fuselage to a monocoque wing. It is of course

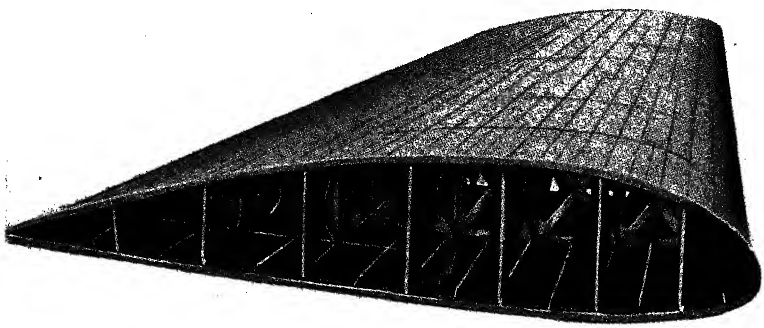


FIG. 122.—Joint in stressed-skin wing. (*Courtesy of Northrop.*)

obvious that the connection may be made through bulkhead rings in the fuselage and bulkheads or heavy ribs in the wing. Figure 123 shows the major portion of such a joint. *A* and *B*

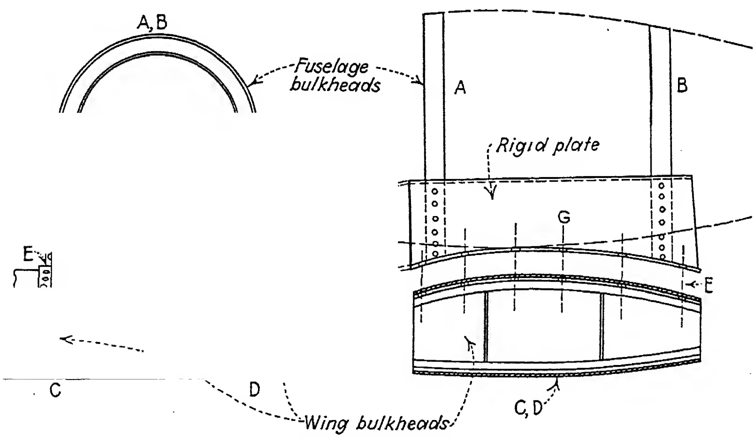


FIG. 123.—Fuselage-wing connection in stressed-skin structure.

are the fuselage bulkhead rings to which is attached a rigid member *G*.

The member *G* is formed to fit the contour of the wing, and by a row of bolts at *E* may be made an integral part of a wing

bulkhead C or D. The fuselage bulkheads, in this connection, must be designed to withstand the stresses in an unbalanced wing-loading condition. Figure 124 shows a slightly different application of the same principle.

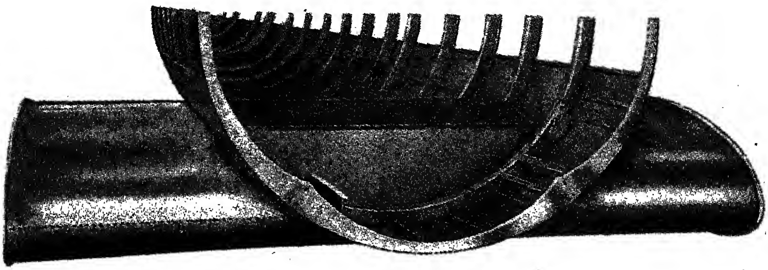


FIG. 124.—Principle of fuselage-wing connection in stressed-skin structures. (*Courtesy of Northrop.*)

Figure 125 shows this principle applied to a semi-monocoque fuselage for attaching a two-spar wing. The rear bulkhead wing fitting is just in front of the door.

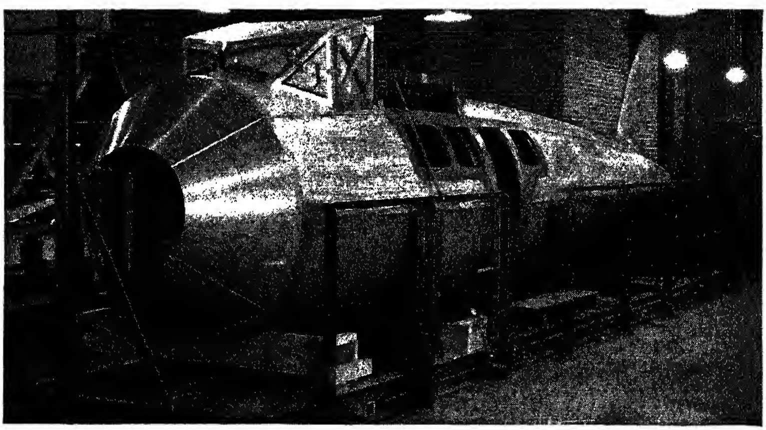


FIG. 125.—Metal-duralumin monocoque fuselage under construction. Fleetster Model, 17, Special. Note rigid rings for wing connection. (*Courtesy of Consolidated Aircraft Corporation.*)

214. Strength and Rigidity of a Bulkhead Ring.—Consider first the case of torsion in the fuselage as from an uneven loading on the right and left span of the wing. The ideal case is that

shown in Fig. 126. The moment PL is calculated from the wing loading. F is therefore determined from

$$Pl = Fb \tag{490}$$

from which F, the design loads for the bolts, for this case, may be found. The free-body diagram of the ring is shown in Fig. 126b,

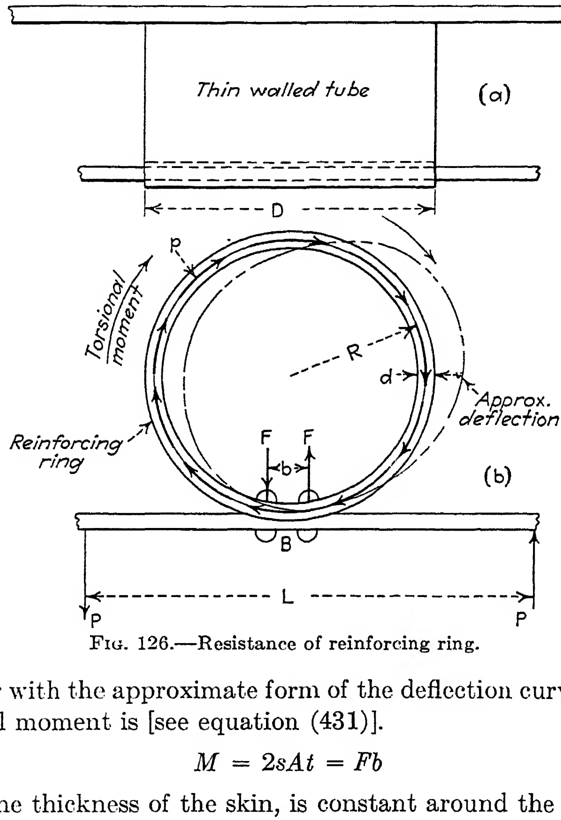


FIG. 126.—Resistance of reinforcing ring.

together with the approximate form of the deflection curve. The torsional moment is [see equation (431)].

$$M = 2sAt = Fb \tag{491}$$

If t, the thickness of the skin, is constant around the ring, the force per unit length is st which is

$$p = st = \frac{M}{2\pi R^2} \tag{492}$$

The maximum bending moment will occur at B, its value being given by equation (490). If strength were the only requirement, it would not be difficult to design the beam to carry the moment at this point. Since, however, *rigidity is of primary importance*, the problem is not so simple.

Since the rigidity is proportional to the width d of the ring, it is desirable to make d as great as possible. A large d, however, is not desirable from the standpoint of space in the passenger cabin, so that a compromise must be arrived at. A reasonable value for d would be about 2 in. to 4 in. for a fuselage of approximately 5 to 7 ft. in diameter.

The material of the ring should be so arranged that the moment of inertia of the cross-sectional area of the ring will be as great as possible. This requirement dictates an I-beam with a thin web as in Fig. 127. (The I-beam may not be selected because of other features, such as ease of production, etc.) For $d = 3$ in., the dimensions for duralumin would be approximately:

$$t_1 = 0.040 \text{ in. to } 0.064 \text{ in;}$$
$$t_2 = \tfrac{1}{16} \text{ in. to } \tfrac{1}{8} \text{ in.}$$

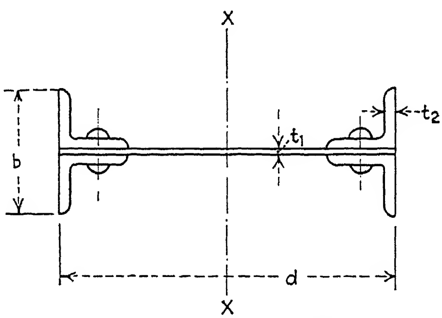


FIG. 127.—Cross section of built-up fuselage reinforcing ring.

The modulus-of-rupture formula [equation (203)] would apply for low stresses.

In general the rigidity requirement would call for a breaking strength much greater than the actual design calls for; therefore *conservative assumptions may be made to simplify the problem in making sure that ample strength exists.*

215. Strength of a Reinforcing Ring.—When a thin-walled tube, reinforced by bulkhead rings such as A and B in Fig. 128, is subjected to a bending moment, the rings are subjected to a vertical compressive load as shown. If we consider (see Fig. 129), that y is the deflection of B below A due to the load W, the vertical force acting on the ring A for a width ds is dF.

Since $$c = y \sin \theta$$
and $$dp = Kcds = Kcrd\theta \qquad (493)$$

in which K is a constant, the value of which is not required in this analysis, and since

$$dF = dp \sin \theta$$

we have

$$dF = Kry \sin^2 \theta d\theta \tag{494}$$

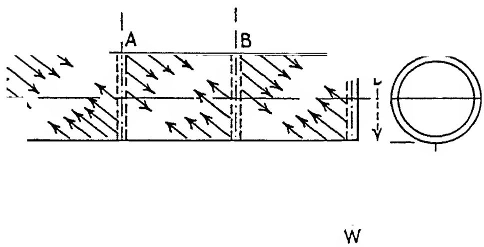


FIG. 128.—Tension lines in thin-walled tubes subjected to bending.

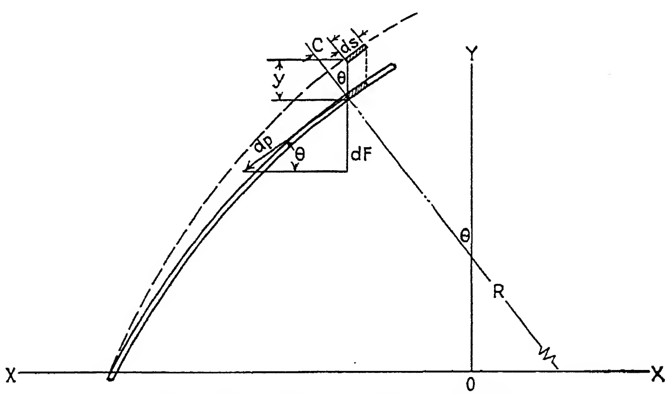


FIG. 129.—Forces on reinforcing ring.

Then

$$W = \int dF = Kry \int_0^{2\pi} \sin^2 \theta d\theta \tag{495}$$

The loading on the circular ring would be therefore of the form shown in Fig. 130, in which dF is given by the equation

$$dF = A \sin^2 \theta d\theta \tag{496}$$

Now it would be possible to extend the analysis to the calculation of the numerical value of the load curve in terms of W,

assuming ideal conditions, especially no wrinkling, and to compute the stresses in the ring as well as the deformation. However, it is doubtful whether this is justifiable in the absence of data on the *required rigidity to preclude wrinkling in the skin.* It is probable that a ring possessing the required rigidity will be many times too strong; hence it appears that for strength calculations conservative approximations of the loading of Fig. 130 and of bending conditions in the ring are sufficient to prove ample strength. A ring built up and loaded with the approximate design load would be desirable for rigidity tests.

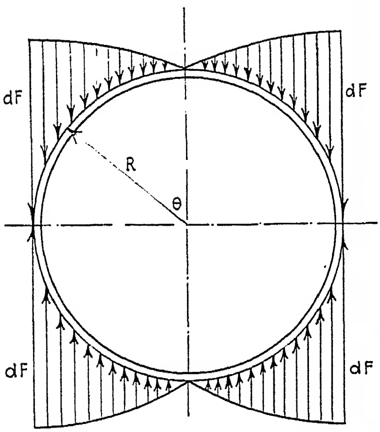


FIG. 130.—Distribution of shear load on a reinforcing ring.

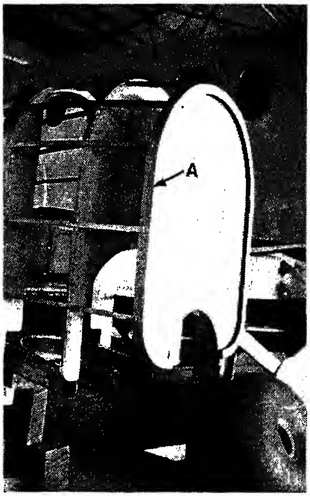


FIG. 131.—Combination firewall and bulkhead fitting for stressed-skin nacelle. (The skin has not been applied.)

216. Attaching Motor Mounts to Monocoque Structures.—The principle discussed in Par. 211 is applied in the design of an engine-mount-fuselage or engine-mount-nacelle fitting. It appears not to be practical to extend a monocoque structure to the mounting ring of the engine because access to the engine auxiliaries requires many openings through the stressed skin; and *openings in stressed skin destroy its efficient use.* Figure 131 shows the bulkheads and bracing framework of a semi-monocoque engine nacelle. The welded-steel tubular engine-mount ring is to be bolted to ring *A* through vibration-absorption material. Figure 132 shows the engine-mount ring in place on the finished

nacelle. Figure 133 shows an engine mount attached to a monocoque fuselage using the basic principle of design.

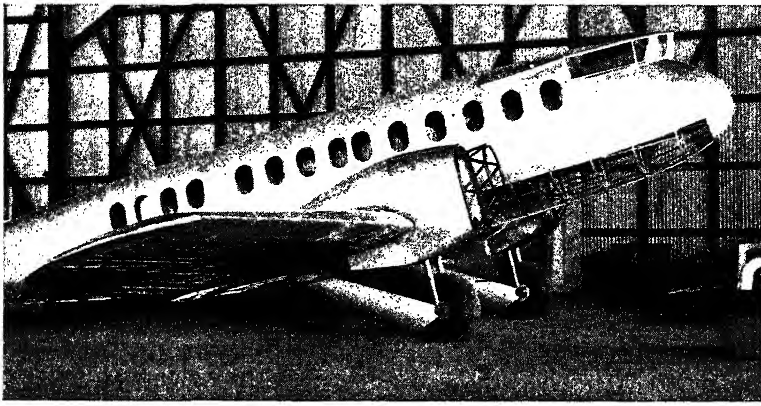


FIG. 132.—Stressed-skin construction.

217. Design of Empennage Connections.—The principle involved in the design of monocoque fuselage connections for vertical fins and horizontal stabilizers is the same as that involved

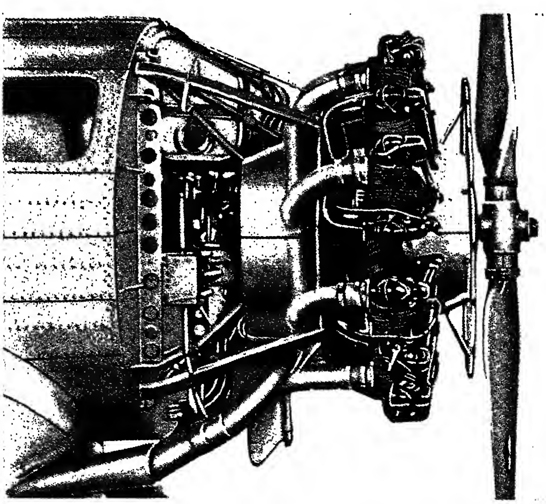


FIG. 133.—Attachment of motor mount to monocoque fuselage. (*Courtesy of Northrop.*)

in the design of a bulkhead for a large thin-walled tube to transmit torsion, bending, and shear to the walls of the tube. Figure

134 illustrates the case. In this figure, B is a bulkhead of the fuselage. A is a fin spar—a built-up I-beam or channel section. The web of the spar and the web of the bulkhead are one continuous sheet of metal. The spar is anchored to the bulkhead ring at C and D so as to carry the design side load on the beam. In covering this connection with the skin of the fuselage, care must be used that no free edges of the skin remain at D where the spar protrudes from the fuselage. The student must bear in mind that the *skin must be homogeneous in every respect to develop its strength.*

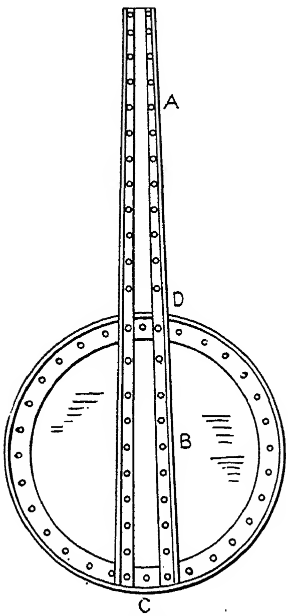


FIG. 134.—Principle of connection of fins and stabilizers to stressed-skin fuselage. The bulkhead B and the spar A are integral.

A variable thickness of skin on the fuselage, as noted from equation (431), is another point to be considered. The strength in shear is proportional to the area of the cross section of the fuselage; hence, since the fuselage approaches a small diameter at the empennage end, the skin must be made thicker to provide for the strength and rigidity. The strength is sometimes improved, as shown in Fig. 135, by slanting the fin spars so that the anchoring bulkheads may be located at points of greater cross-sectional area in the fuselage.

These same principles, of course, are also used in the design of stabilizer connections.

218. Design of Chassis-monocoque-wing Connections.—The principle involved in this design problem is the same as that involved in the design of a fitting to transmit and distribute a concentrated load applied perpendicularly to the axis of the tube. In this case, two wing bulkheads will be required to provide for the side-load condition. Figure 136 shows a wing connection of this type. (The holes in the bulkhead or rib are hand holes.) *Lightening holes in a bulkhead or rib, designed for rigidity in the transmission of stresses due to concentrated loads, seem to be of doubtful value.*

219. Application of Stringers.—To increase the axial compressive strength of large thin-walled tubes, reinforcing *stringers*

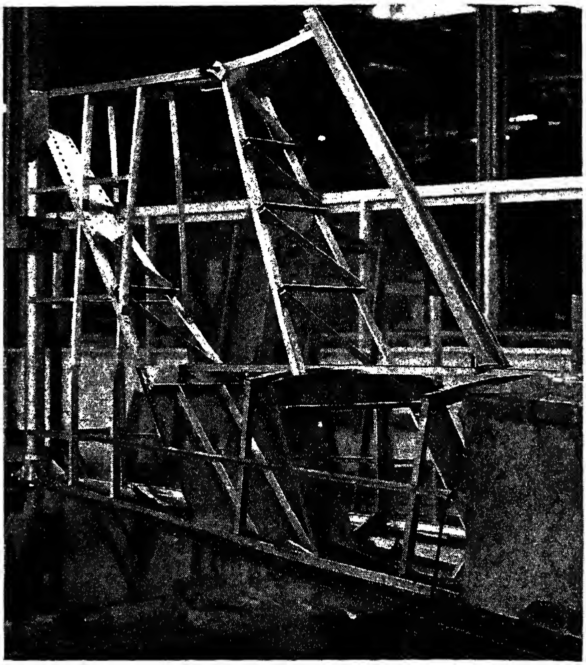


Fig. 135.—Fuselage-fin connection. (*Courtesy of Consolidated Aircraft Corporation.*)

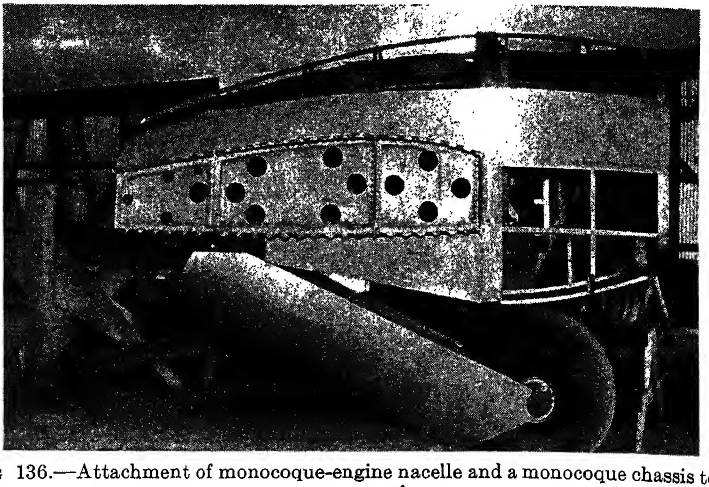


Fig 136.—Attachment of monocoque-engine nacelle and a monocoque chassis to a monocoque wing.

parallel to the axis of the tube are used (see Figs. 104 and 137). In calculating the strength, in compression, of the combination of stringers and tube walls, it should be borne in mind that *a failure of the weaker of the two may cause a premature failure in the stronger.* In general the combined strength is the product

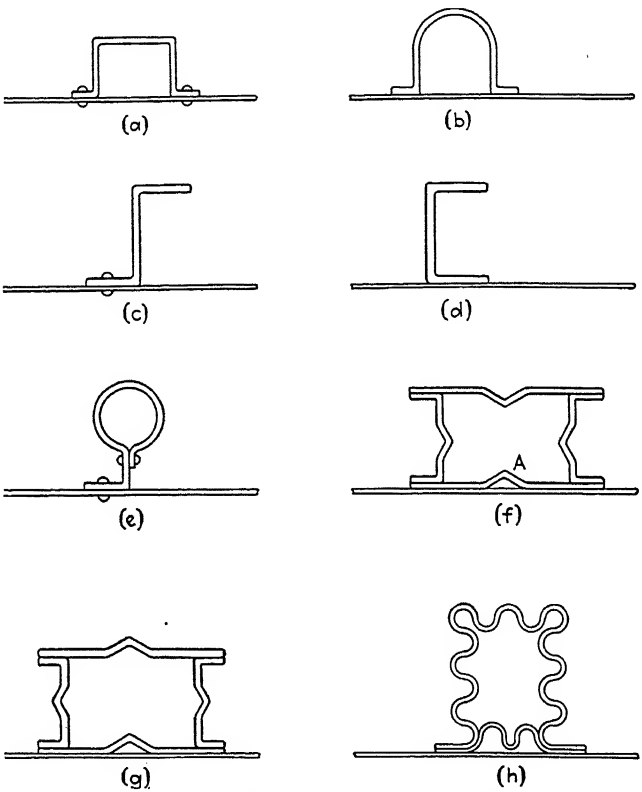


FIG. 137.—Longitudinal stiffeners or stringers.

of the unit stress at failure of the weakest element and the area of the total cross section bearing the compressive load. Failure may occur in any one or more of the following ways:

1. Wrinkling of the sheet-metal skin.
2. Buckling of the stringer as a column in the direction of the radius of curvature of the sheet.
3. Buckling of the stringer in the direction perpendicular to the radius of curvature (twisting of the stringer).
4. Local wrinkling of the stringer.

220. Types of Stringers.—Let us consider the advantages and disadvantages of the stringers in Fig. 137 with reference to the types of failure listed in the last paragraph. The *Z*-section *c* and the channel section *d*, while they may be attached readily to the skin, have free edges which readily wrinkle, permitting the stringer to fail sideways. The bulb-angle section *e*, while containing no free edges, and while quite stiff against buckling normal to the sheet, may buckle in torsion unless well braced. The hat section *a*, while easy to attach and possessing great resistance in buckling in planes both normal and parallel to the plate, contains considerable flat-surface area, which for thin material will wrinkle readily. Section *b* has the same characteristics, but on account of the curvature of the section is resistant to wrinkling. Section *f* is quite generally used for stainless steel, since it is easily fabricated. The *v*-notches in the sides strengthen the flat sides against wrinkling. There are free edges, but these are limited in width. If the skin is thin in comparison to the thickness of the plate of which the stringer is made, the stringer should be made complete by the addition of strip *A* between the stringer and the skin.

Section *g* contains the same amount of material as section *f* but its radius of gyration is greater, improving the section from a strength standpoint. *Section h from a strength-weight-ratio standpoint with respect to wrinkling, buckling, and twisting is the best of the sections.*

221. Buckling of Stringers as Columns.—Closed tubes, such as in Fig. 137 *a*, *b*, *g*, *f*, and *h*, used as columns are subject to calculations by the column formulas of Euler and Johnson. However, sections such as the channel section with free edges will fail because of the instability of the free edges, and hence they are not subject to these calculations. Special curves and methods are required for each such type of section.

When a closed section is used as a stringer, the stringer is restrained from buckling in three directions by the skin—parallel, each way, to the sheet, and outward from the center of the tube. The stringer however may buckle inward, carrying the skin with it. It is thus apparent, that *a small eccentricity in the loading on the stringer to give it a tendency to fail outward is desirable.* This may be done at the bulkhead rings by allowing the ring the right of way next to the skin at the crossing of the stringer and ring (Fig. 138), and/or supplying gusset plates.

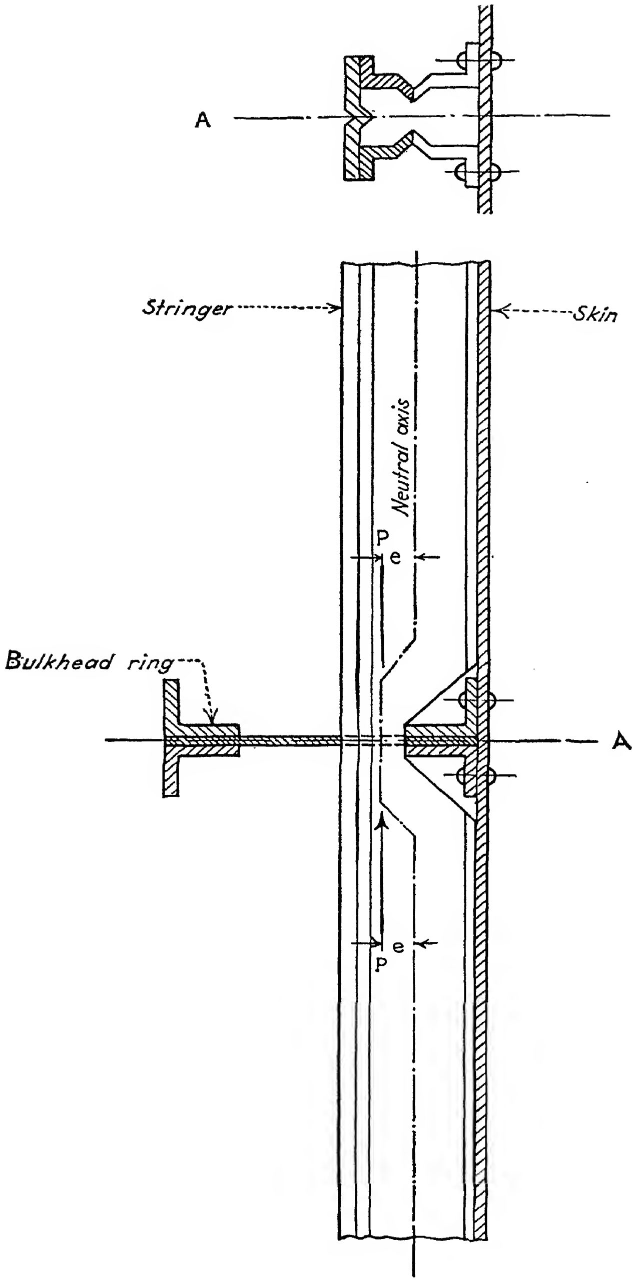


FIG. 138.—Eccentricity in stringer.

The moment tendency to hold the stringer against the skin is Pe.

222. Mutual Support of Stringers and Sheet.—In a stringer-sheet combination, the stringers support the sheet metal against wrinkling and the sheet metal supports the stringers against buckling. Three methods are obvious in the analysis of the strength of such a combination.

1. Assumption of independent action of stringer and sheet.
2. Assumption that the load carried is the sum of the load carried by the stringer when tested alone plus the load carried by an effective width of the sheet adjacent to the stringer. The stress in the effective width is assumed to be the same as the stress in the stringer.
3. Assumption that a combination of an effective width of sheet and the stringer behaves as a column. The radius of gyration and area of the column are calculated for the combination.

The first method, (1), seems to be most logical for a tube of small radius and fairly thick skin, in which the critical stress computed for the skin is approximately equal to the critical stress for the stringer. For example, if we consider a duralumin tube 1 ft. in diameter and skin 0.05 in. in thickness, we compute from equation (476) the allowable stress as follows:

$$f = 0.12 \times 10{,}000{,}000 \times \frac{0.05}{6} = 10{,}000 \text{ lb. per square inch}$$

If we assume stringers with radius of gyration of 0.1 in. and a length of 10 in. between bulkhead rings, we have an L/ρ of 100, which gives us

$$\frac{P}{A} = \frac{\pi^2 E}{(L/\rho)^2} = \frac{9.86 \times 10{,}000{,}000}{(100)^2} = 9{,}860 \text{ lb. per square inch}$$

It appears that in this case we can expect very little if any mutual support between stringers and skin, because a slight wrinkle in the skin between the stringers would immediately influence the stability of the stringers. A conservative and practical fiber stress to assume in this case, in the absence of specific data on the actual sections, would be the lower stress of 9,860 lb. per square inch.

223. Stringers for Large Tubes.—As the size of a tube, such as a fuselage or wing section, is increased, the advantages of monocoque construction seem to decrease. Above this limit a semi-monocoque structure with stringers and bulkhead rings seems to be desirable, with the load being carried in unison by

the stringers and skin. However, a further increase in size seems to lessen the advantage of the skin in carrying stress, so that, of necessity, we revert to the frame type of structure with the skin being used to carry tension and shear only (with no compression), to support the stringers, and to cover the frame.

For such large tubes it seems desirable to use as thin skin as possible for covering and shear, and to require the stringers, exclusively, to carry the compressive load. The stringers then may be designed approximately and conservatively (for closed sections) by column formulas.

224. Flat Sheet under Edge Compression.—Data are available (see Reference 12) on the compressive strength of flat sheets of stainless iron, duralumin, monel metal, and nickel. The average physical properties of the materials are shown in Table XIX.

TABLE XIX.—PHYSICAL PROPERTIES OF MATERIALS
(For Figs. 139, 140, 141, and 142)

Material	Tensile strength, lb./sq. in.	Yield point, lb./sq. in.	Modulus of elasticity, lb./sq. in.
Stainless iron	80,000	45,000	29,000,000
Duralumin...	60,000	40,000	10,000,000
Monel metal.	80,000	30,000	24,000,000
Nickel......	70,000	35,000	28,000,000

Figures 139, 140, 141, and 142 show curves for the strength of the plates under edge compression, with the vertical edges of the plate held from lateral motion by *v*-notch grooves.

A study of the curves will show that the load curves rise rather rapidly, almost in proportion to the width for a few inches of width. *It appears that the width at which the curve deviates appreciably from a straight line should be the optimum spacing of the stringers for flat sheets or for curved sheets with very small curvature.* If stringers are spaced farther apart than the optimum, then the central portion of the sheet between the stringers carries very little of the load as indicated by the tests. If we take the optimum width as b, then for spacing wider than b we may assume that for a width b, bisected by the stringer, the load as indicated by the curves for width b is carried by the flat sheet. We may call the width b the *effective width,* which may be

assumed to be acting with the stringer considered as a strut. This effective width would then apply to points (2) and (3) of the analysis in Par. 222. It must be borne in mind that the design stress must not be higher than the stress (pounds per square inch) computed for width b from the figures.

225. Stringer-stressed-skin Combination.—If the data of the last paragraph are used for computing the strength of a stringer-stressed-skin combination for a slightly curved surface, on the assumption that the curved sheet will be only a little stronger than the flat sheet, care must be used in computing the moment of inertia of the cross-sectional area of the combination. The

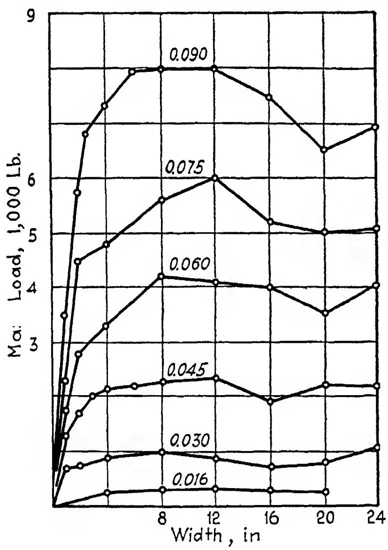


FIG. 139.—Maximum load for duralumin plates 24 in. long in direction of loading. (*Courtesy of N. A. C. A.*)

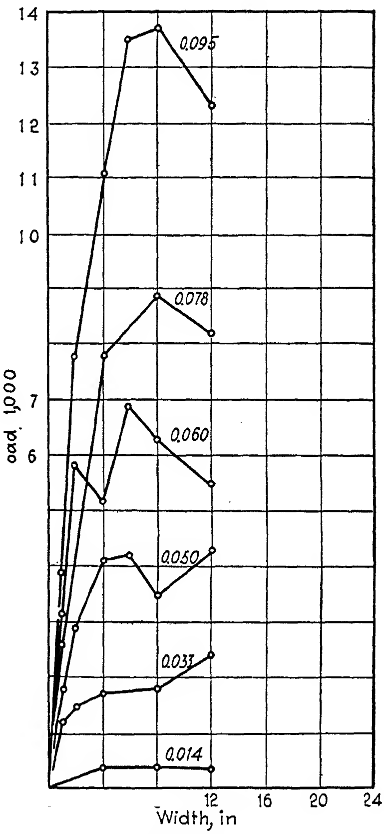


FIG. 140.—Maximum load for stainless iron plates 24 in. long in direction of loading. (*Courtesy of N. A. C. A.*)

moment of inertia of the combination for the curved sheet will be less than for the flat sheet. This is apparent from Fig. 143. While the difference in strength experimentally and theoretically is small and may be neglected, the student must bear in mind that the condition exists.

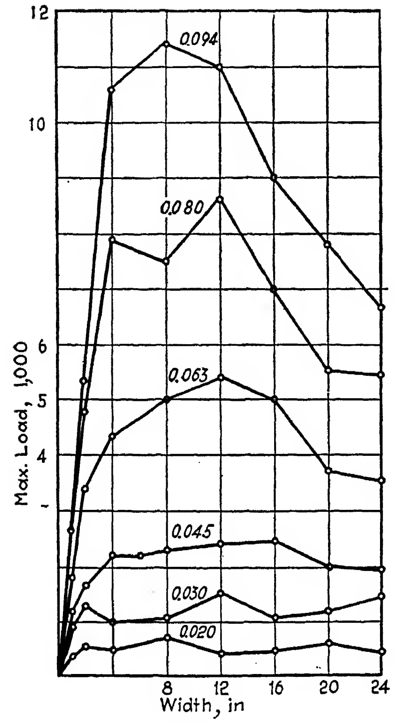


FIG. 141.—Maximum load for monel metal plates 24 in. long in direction of loading. (*Courtesy of N. A. C. A.*)

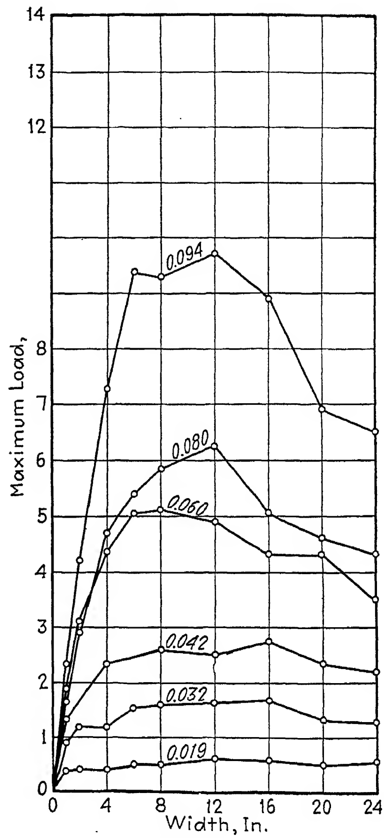


FIG. 142.—Maximum load for nickel plates 24 in. long in direction of loading. (*Courtesy of N. A. C. A.*)

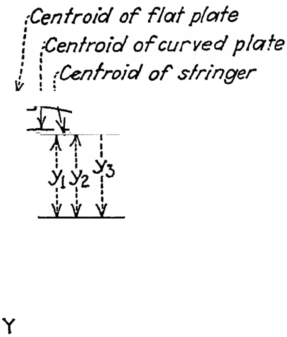


FIG. 143.—Combination of stringer and plate.

226. Windows and Doors for Stressed-skin Fuselage.—The principle to be used in the design of window and door openings in a stressed-skin fuselage is as follows: *The bracing and reinforcement about the windows and doors must be such that the distortion of the ring about the opening will not be greater than the distortion of a similarly placed ring on the sheet metal if the opening did not exist.* This would indicate that openings should be placed where the stress is the lowest, if a choice is allowed.

This specified condition, of course, is the theoretical ideal which may never be attained. The student should bear this principle in mind, and approach the problem from the standpoint of rigidity first, and strength second.

In general a door in a semi-monocoque fuselage requires several stringers to be cut. The loads in the stringers, obviously, must be transmitted around the door without undue distortion of the

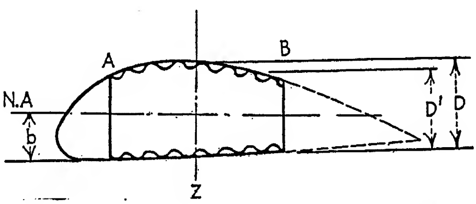


FIG. 144.—Elements of a stressed-skin wing.

fuselage at the door section. If undue distortion is permitted, high stresses will be introduced at other points of the structure. The design then becomes the design of a ring beam around the door subjected to parallel compressive forces, with the special requirement that the ring should not be distorted more than would be the case if the door were not cut out of the surface.

227. Design of a Stressed-skin Wing Structure.—The major phases of such a design are:

1. Determination of the applied loads.
2. Determination of the elastic axis.
3. Design of flanges.
4. Design of webs.
5. Design of bulkheads (ribs).

Let us now consider these last three items in detail.

228. Calculation of Axial Load in Flanges.—For a stressed-skin wing in which the center portion of the wing, A to B (Fig. 144), is the supporting spar, it is probably sufficiently accurate to

calculate the load in the flanges for the preliminary design as follows:

Let

M_x = bending moment to which the wing is subjected at station x.
P = total axial load on a flange.
D' = average depth of spar.

Then

$$P = \frac{M_x}{D'} \tag{497}$$

A more accurate method for final design, which will apply to a stressed-skin wing with any number of spars, is as follows: First, find the neutral axis by the formula

$$b = \frac{\Sigma a_1 y_1 + a_2 y_2 + a_3 y_3 + \cdots}{\text{Area of supporting material}} \tag{498}$$

in which a_1, a_2, etc., are small increments of area, and y_1, y_2, etc., are distances from the reference line to the increment of area.

Second, compute the moment of inertia of the area of the supporting material about this axis by the formula

$$I = \Sigma a_1 v_1^2 + a_2 v_2^2 + a_3 v_3^2 + \cdots \tag{499}$$

in which v is the distance of the small increment of area from the neutral axis.

Then, third, if M_x is the bending moment at the section x, the fiber stress f in pounds per square inch is computed from

$$f = \frac{M_x v}{I} \tag{500}$$

To find the approximate load on the flange of a spar, take v the average for the flange and compute

$$P = fA = \frac{M_x \bar{v}}{I} A \tag{501}$$

in which P is the total axial load on the flange being considered and A is the area of the cross section of the flange.

This method may be used to find the loading due to the chord component of air load by finding the neutral axis Z-Z.

No twisting of the wing, of course, is assumed in these methods. This is a reasonable assumption if the shell cover is continuous around the periphery of the cross section.

The *allowable load* in the flanges is important. This is a function of the structural characteristics of the wing. In general the allowable load in column action is the design allowable load.

If data are not available on the particular design, only very approximate prediction of strength can be made unless experiments are resorted to.

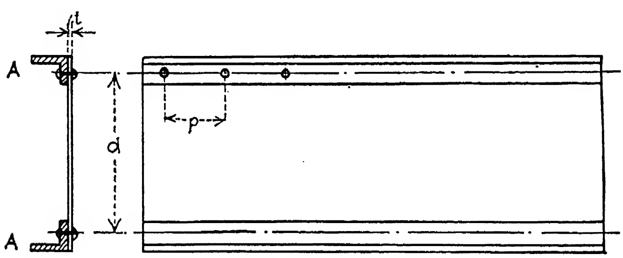


FIG. 145.—Elements of a thin web.

229. Theory of Thin Webs in Deep Beams.—The torsional strength as well as the bending strength of a single-spar box wing beam depends upon the wrinkling of the sides and the buckling of the edges. The theory of this type of failure is exemplified in thin-web deep beams.

The longitudinal unit shear in the web of a beam (see Par. 139) is

$$s = \frac{F}{It}\bar{v}A \tag{502}$$

in which

s = shear stress, pounds per square inch.
F = total shear on the beam, at the section being considered, pounds = V.
t = thickness of the web, inches.
$\bar{v}A$ = the static (area) moment of the area above or below the point where the shear is being determined = Q.

If the web is thin, as shown in Fig. 145, we may assume A to be the area of a flange and v to be half the effective depth of the

beam d. I, in this case, would be very nearly $2A(d/2)^2$. Thus, for a thin-webbed deep beam, equation (502) may be written

$$s = \frac{F\frac{d}{2}A}{2A(d/2)^2 t} - \frac{}{dt} \tag{503}$$

Now, as illustrated in Fig. 146, the shear F is the resultant of two components whose lines of action are at right angles to each other, the compressive component P and the tensional com-

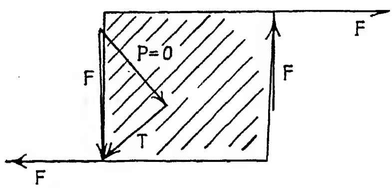


FIG. 146.—Stress in thin web of deep beam.

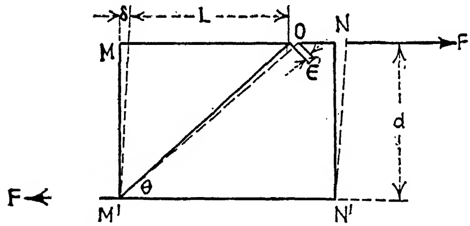


FIG. 147.—Direction of tension lines in a thin web.

ponent T. In Fig. 147 is represented a portion of the web, between the flanges MN and $M'N'$, subjected to the shear force F. The shearing force causes a shearing deformation δ. We will assume the maximum tensile stress f to be acting along the line $M'O$, making an angle θ with the lower flange. The length of $M'O$ is $\frac{d}{\sin\theta}$. The tensile deformation is ϵ, which may be written, approximately,

$$\delta \cos\theta \tag{504}$$

The unit strain is

$$\frac{}{\frac{d}{\sin\theta}} = \frac{}{d}\sin\theta\cos \tag{505}$$

The unit tensile stress is f,

$$f = \frac{\delta E}{d} \sin \theta \cos \theta \tag{506}$$

230. Angle of Tension Lines.—To find the maximum value of f, we equate the derivative of equation (506) with respect to θ to zero. Thus

$$\frac{df}{d\theta} = -\frac{\delta E}{d}(\sin^2 \theta - \cos^2 \theta) = 0 \tag{507}$$

from which

$$\sin^2 \theta = \cos^2 \theta, \qquad \text{or} \qquad \tan^2 \theta = 1$$

or

$$\theta = 45 \text{ degrees} \tag{508}$$

The angle θ is generally, from experimental data, assumed to be 42 degrees. If the web is too thin to take an appreciable amount of compression, only half of the resisting shear force is available to resist deformation. This may be introduced into our formulas of rigidity by substituting

$$\frac{t}{2} \text{ for } t \tag{509}$$

231. Tensile Stress in Thin Web.—The unit tensile stress developed in a thin web in which the compressive stress is negligible is determined as follows:

The shear load on a length ΔL along the line of intersection of web and flange is stated as

$$\text{shear load} = ts\Delta L \tag{510}$$

The tensile load T at θ degrees to the line of action of the shear load is expressed as

$$T = \frac{ts(\Delta L)}{\cos \theta} \tag{511}$$

The cross-sectional area subjected to the tensile load T is

$$t(\Delta L) \sin \theta \tag{512}$$

Hence the unit tensile stress is obtained:

$$f = \frac{ts(\Delta L)}{\cos \theta} \frac{1}{t(\Delta L) \sin \theta} = \frac{s}{\sin \theta \cos \theta}$$
$$= \frac{F}{td \sin \theta \cos \theta} \tag{513}$$

If θ is taken as 45 degrees, $\cos \theta \sin \theta = 0.5$; hence $f = 2s$.

232. Load on Vertical Compression Member.—If the beam is uniform in depth, and the vertical stiffeners or compression members may be assumed uniformly spaced for increments of

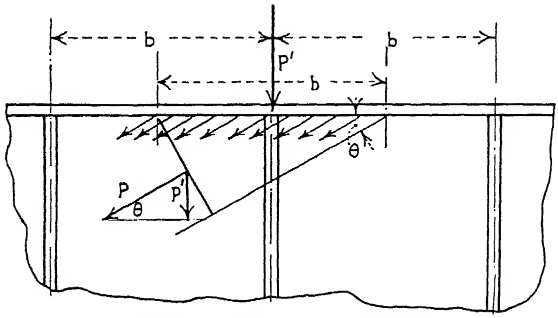


FIG. 148.—Loads on web stiffeners.

length, which is generally the case, the total or design load of the vertical member may be determined as follows (see Fig. 148):

$$P = ftb \sin \theta \tag{514}$$

and

$$P' = P \sin \theta = ftb \sin^2 \theta \tag{515}$$

Since from (513),

$$f = \frac{s}{\sin \theta \cos \theta}$$

then

$$\begin{aligned} P' &= stb \tan \theta \\ &= stb \text{ (for } \theta = 45 \text{ degrees)} \\ &= 0.9stb \text{ (for } \theta = 42 \text{ degrees)} \end{aligned} \tag{516}$$

and since by (503),

$$st = F/d$$

We have

$$P' = \frac{F}{d}b \tan \theta = \frac{F}{d}b \text{ (for } \theta = 45 \text{ degrees)} \tag{517}$$

If the compression struts are riveted to the web, failure may not occur in buckling as a column, as the tension in the web will prevent lateral buckling.

The rivets through the stiffeners and chord members should be designed for the total load on the stiffeners as the rivets through the web serve only to prevent lateral buckling of the stiffeners.

It appears practical to assume an allowable stress of P/A not over the elastic limit of the material.

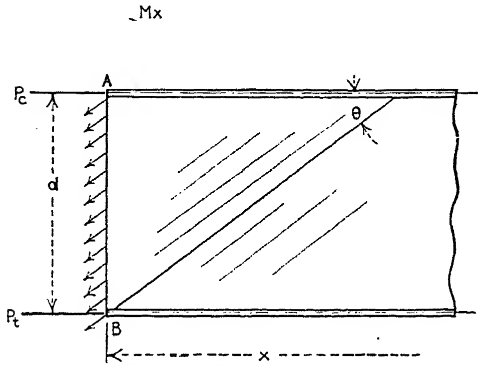


FIG. 149.—Loads in flanges of a thin-webbed beam.

233. Load in the Flange Members.—With reference to the free-body diagram (Fig. 149), we have a beam subjected at x to a bending moment M_x. Writing $\Sigma M_A = 0$, we have

$$M_x = P_t d + ft\frac{d^2}{2}\cos^2\theta \tag{518}$$

from which

$$P_t = \frac{M_x}{d} - \frac{ftd}{2}\cos^2\theta \tag{519}$$

Since $f = \dfrac{s}{\sin\theta\cos\theta}$, and $s = \dfrac{F}{dt}$, we have

$$P_t = \frac{M_x}{d} - \frac{F}{2}\cot\theta \tag{520}$$

The design load will probably be the compression load. In this case

$$P_c = \frac{M_x}{d} + \frac{F}{2}\cot \tag{521}$$

234. Sagging of Flange Members.—The tension field of the web of a deep and thin-webbed beam has a tendency to cause the

flanges to sag as noted in Fig. 150. This throws an undue stress on the web and the riveting of the web at the strut (or stiffener) sections of the flange. The success of a beam of this type requires that the tension field in the webs at the flanges be

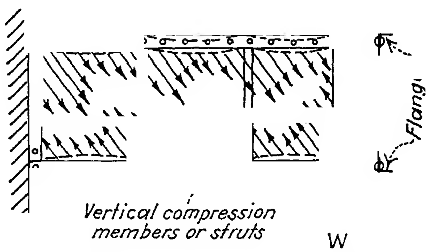


FIG. 150.—Shear in a thin web, and sag of flanges.

fairly uniform. This requires that the flange be as stiff as possible in resisting sagging. Figure 151 shows one method of providing this stiffness. The strip A is inserted to provide a large

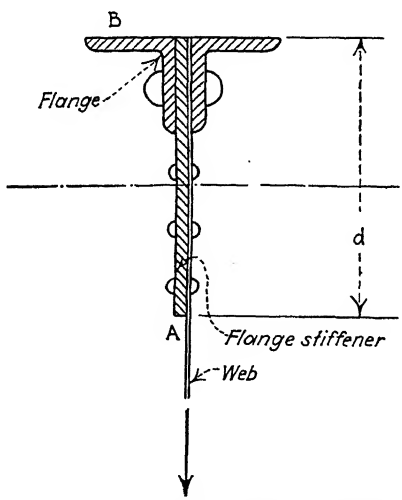


FIG. 151.—Stiffening a flange against sagging.

moment of inertia about the x-x axis, and to provide sufficient rivet area for riveting the thin web to the flange.

235. Design of Flanges.—In general these flanges may be considered as beam columns with the following characteristics:

1. Uniform moment of inertia of cross-sectional area between compression stiffeners.

2. Column with axial load as computed from equation (521).

3. Beam with a total bending load of P' as calculated from equation (517).

4. Column supported against buckling sideways. If the beam is sufficiently braced against sagging, a very high compressive stress may be carried. The beam is designed for the total primary bending and compressive load in sagging plus the secondary stress as a beam column (see Sec. III, Chap. XI).

236. Practical Design of Rivets for Web.

1. *Rivets Attaching Web to Flanges.*—The load per rivet is

$$P_r = ftp \sin \theta \tag{522}$$

in which f is the fiber stress in tension, t is the thickness of the web, and p is the pitch of the rivets. Substituting the value of f from equation (513), we have

$$P_r = \frac{Fp}{d \cos \theta} \tag{523}$$

If θ is taken as 42 degrees, equation (522) becomes

$$P_r = 0.669ftp \tag{524}$$

and equation (523) becomes (see Fig. 152),

$$P_r = 1.338F\frac{p}{d} \tag{525}$$

If double rows of rivets are used, p is one-half of the rivet pitch for each row.

2. *Rivets through Web Splice.*—In this case the pitch of the rivets p is measured vertically so that

$$P_r = ftp \cos \theta = \frac{Fp}{d \sin \theta} \tag{526}$$

Thus, if θ be taken as 42 degrees,

$$P_r = 0.743ftp = 1.486F\frac{p}{d} \tag{527}$$

3. *Loads on Rivets between Caps.*—When caps are riveted on to the chord member, the loads on rivets are given by the following formulas:

$$S_r = \frac{F}{d}\frac{A_c}{A_t} \quad \text{or} \quad S_r = st\frac{A_c}{A_t} \tag{528}$$

in which

S_r = load per inch along beam between cap and chord member.
A_c = cross-sectional area of cap.
A_t = total cross-sectional area of chord member including cap.

$$P_r = pS_r = b\frac{F}{d}\frac{A_c}{A_t} \qquad \text{or} \qquad P_r = pts\frac{A_c}{A_t} \tag{529}$$

in which

p = effective rivet pitch.
P_r = load on each rivet.

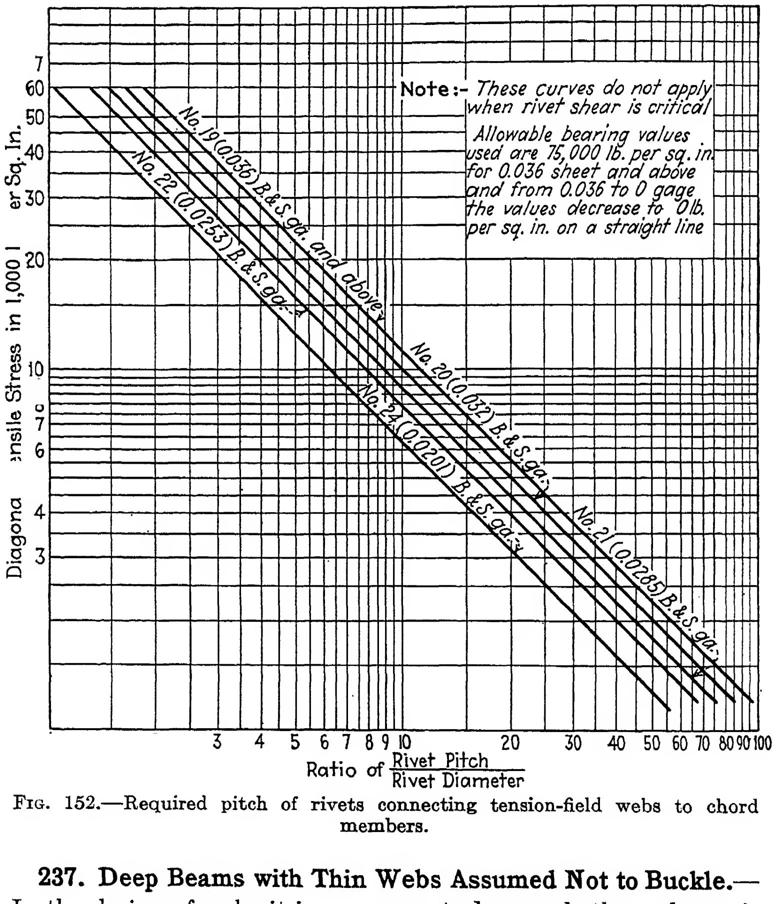


FIG. 152.—Required pitch of rivets connecting tension-field webs to chord members.

237. Deep Beams with Thin Webs Assumed Not to Buckle.—In the design of webs it is necessary to know whether a beam is in the tension-field range or whether it is shear resistant. A good

criterion, based on the law of similarity, is the index value $\sqrt{F}/d$, where F is the applied shear and d is the depth of the beam. If this index value is less then 8.32, the web should undoubtedly be designed as a tension-field web. On the other hand, if the index value is above 12.5, a shear-resistant web is in order. Since the transition from tension-field to shear-resistant web is at no definite value, a check of both types of design should be made to determine the most economical structure. This study should involve both webs and stiffeners. It is well to remember that a plate girder is usually less costly in production than a tension-field web-type beam.

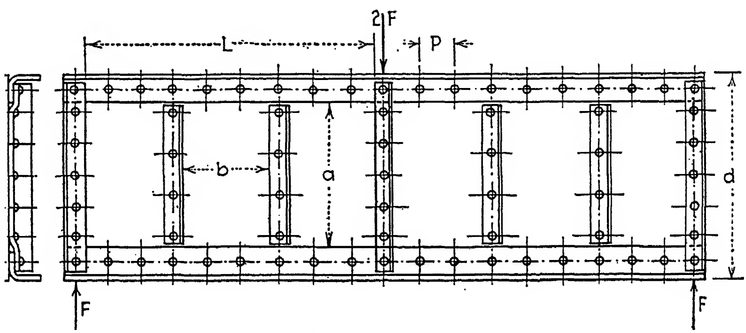


FIG. 153.—Riveting a thin web in a deep beam.

The critical shear which produces buckling of the plate web can be represented by the formulas[1]

$$s = s_a - \overline{3.64KE}\backslash\bar{t} \tag{530}$$

or if b/t is greater than $[1.82KE/s_a]^{1/2}$

$$s = \frac{0.91KE}{(b/t)^2} \tag{531}$$

where

s_a = ultimate shear allowable

b = distance between stiffeners; if, as in the case of a shallow beam, intermediate stiffeners can be eliminated, b becomes a.

t = web thickness.

[1] TIMOSHENKO, S., and J. M. LESSELS, "Applied Elasticity," Westinghouse Technical Night School Press, East Pittsburgh, Pennsylvania, 1925.

The coefficient K depends upon the ratio of a/b. If the beam requires no intermediate stiffeners, a/b becomes L/a (see Fig. 153); as L/a increases, K approaches the value 5.7($L/a = \infty$). The curve (Fig. 154) indicates the relation between L/a and K.

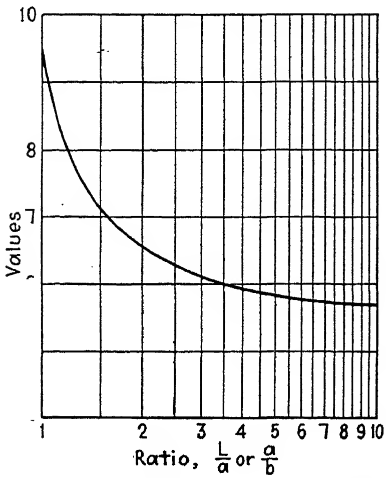


FIG. 154.—Constants for calculating allowable shear stress.

238. Design of Stiffeners for Non-buckling Web.—If the distance between chord sections is too large (greater than 50 to 60 times the web thickness), intermediate stiffeners must be used. The size of these stiffeners can be determined from the equation[1]

$$I_{st} \quad \frac{2.29d_{st}}{t}\left(\frac{Fd}{33E}\right)^{\frac{4}{3}} \tag{532}$$

where

I_{st} = moment of inertia of stiffener.
d_{st} = rivet center line distance between stiffeners.
t = web thickness.
F = applied shear load.
d = total depth of beam.

239. Chord Loads, Beams with Stiff Webs.—The flexural stress in a plate girder is determined by the formula

$$f = \frac{Mc}{I} \tag{533}$$

Since the web of a plate girder is subjected to flexural stress, it is advisable to check the web, at the junction of the web with the flanges, for a combined flexural and shear stress.

$$F_{c(\text{max.})} \tag{534}$$

[1] Refer to paper presented by Herbert Wagner at the Fourth National Aeronautic Meeting of the A. S. M. E. at Dayton, Ohio, May, 1930.

where

F_c = principal buckling stress.

f = flexural stress [(equation (533)].

s = shear stress.

This combined stress may in certain cases be higher than the extreme fiber stress.

Concentrated load stiffeners must be designed to take the total applied load and should be attached to both web and chord section. A sufficient number of rivets should be used to transmit the entire load into the web.

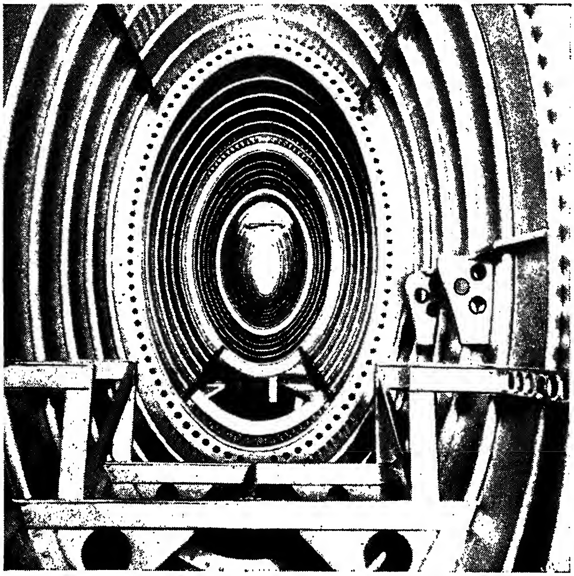


FIG. 155.—Model XB-10. Fuselage interior construction. Note tension and compression corrugations in top and bottom. (*Courtesy of Glenn L. Martin Company.*)

Intermediate stiffeners need not be attached to the chord section except in special cases such as a keelson.

The shear load per inch between the web and the chord angles is given by the equation

$$S = \frac{FQ}{I} \tag{535}$$

where Q is the static moment of one set of chord sections about the neutral axis of the beam.

The load on each rivet $= P_r = ps$, where p is the effective pitch.

The above formula may also be applied to rivets connecting caps to chord angles. The symbol Q will then be the static moment of the cap.

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CHAPTER XIV

PROBLEMS IN CANTILEVER WING DESIGN

240. Applied Loads.—The determination of the distribution of air loading on a cantilever wing for the various assumed or required design conditions is a problem in aerodynamics. If the engineer is confident that government requirements, in this respect, are sufficiently accurate for design purposes, he may proceed with load calculation as specified in the requirements.

In using required loading conditions for design purpose, the student should consider carefully the following problems:

1. *The distribution of the air load on a twisted wing.*—In such a wing, the angle of attack varies with the semi-span. Wings are sometimes designed twisted so that all sections experience a maximum ratio of lift to drag at the same time or so that all sections experience a maximum lift at the same time. It is apparent, in this case, that the air-load distribution with respect to the semi-span is a function of the angle of attack at the section under consideration.

2. *The distribution of air load on a wing, the cross section of which varies with the semi-span.*—Such a wing is one that varies in chord and ordinate-chord ratio. The wing will have a thick cross section at the root and a thin cross section at the tip.

3. *The redistribution of the air load on the wing because of twisting in the wing.*—It has been the usual practice to assume wings to twist, in calculating the distribution of loads on wing spars, without taking into consideration the aerodynamic effect of such twisting. It is suggested that where the twist of a wing is appreciable, a careful investigation concerning its flutter tendencies should be carried out.

241. Resolution of Wing Loading into a Force and a Couple.—It appears more logical and simple to analyze a cantilever wing from the standpoint of pure bending and pure torsion, and then to add the resulting stresses algebraically, than to analyze it from the standpoint of spar-distribution loading based on spar rigidity in bending. The old method is based on the assumption that the

rigidity of the wing in torsion depends solely on the spar rigidity in bending. As a matter of fact the *spars of modern airplane wings of the cantilever type contribute, in general, only a very small percentage of the torsional rigidity of the wing.*

The principle involved in the proposed method of analysis is that of the resolution of a force into a force and a couple. For example, as noted in Fig. 156, the load W of a, applied x-distance in front of O, may be resolved into a load W through O and a couple Wx about O. Now, if, when the load W is applied at O, no twisting in the wing results, the wing may be analyzed for stresses in pure bending caused by the load W, and for stresses in pure torsion caused by the couple Wx. The algebraic sum of the stresses for the two conditions will then be the net design stresses.

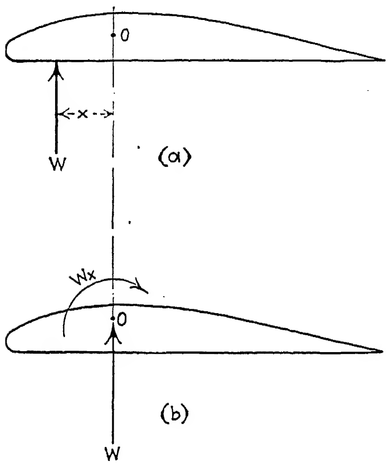

FIG. 156.—Resolution of wing loading into load through, and torque about, elastic axis.

242. Elastic Axis.—*The elastic axis of a cantilever wing is a line approximately parallel to the spars, along which a load may be applied without inducing a twist in the wing.* Also we may define it as the axis about which the wing rotates when subjected to a couple. This axis, the student will note, is that indicated by O in Fig. 156.

The elastic axis of a wing may be determined quite readily by experimental methods. In simple cases, the axis may be determined by theoretical methods for stresses below the elastic limit of the material. If, however, the wing is quite rigid in torsion, as, for example, a "stressed-skin wing,"—one which carries the flying load in the skin—very little error will be involved in determining the elastic axis approximately. The student may note that, for a two-spar wing, the location of the elastic axis is not necessary, except for computing the couple, since, when such a wing is subjected to a couple, the spar loads form a couple. The spar loadings induced by the couple will therefore be equal and opposite. It follows, of course, from this statement that the bending moments are equal and opposite.

243. Location of Elastic Axis.—To fix ideas and to illustrate a method of analysis, let us consider the simple case shown in Fig. 157.

The weight W is being carried by three elastic springs of spring constants k_1, k_2, and k_3. We are to find x so that

$$y_1 = y_2 = y_3 \tag{536}$$

Letting w_1, w_2, and w_3 represent the loads carried respectively by each spring, we have

$$\Sigma F_y = 0$$

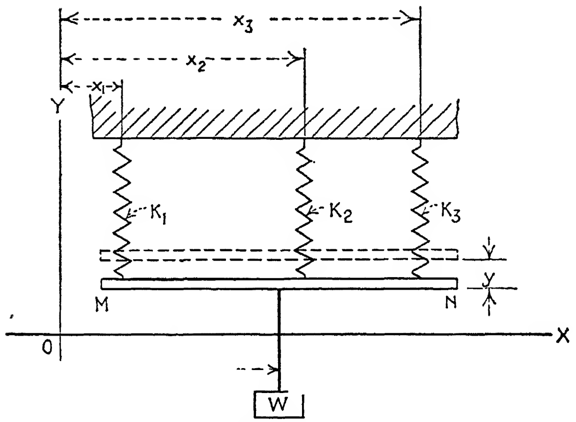

FIG. 157.—Elements of elastic axis.

or

$$W = w_1 + w_2 + w_3 \tag{537}$$

Also,

$$\Sigma M_0 = 0$$

or

$$W\underline{x} = w_1x_1 + w_2x_2 + w_3x_3 \tag{538}$$

From equation (536) we find

$$y_1 = \frac{w_1}{k_1} = y_2 = \frac{w_2}{k_2} = y_3 = \frac{w_3}{k_3} \tag{539}$$

Now three equations of the type represented by equations (537), (538), and (539) are sufficient and necessary to solve for $\underline{x}$, locating the elastic axis. From equation (539) we have

$$w_2 = \frac{k_2}{k_1}w_1 \qquad \text{and} \qquad w_3 = \frac{k_3w_1}{k_1} \tag{540}$$

Substituting equations (540) and (537) in (538), we have

$$\underline{x} = \frac{w_1x_1 + \dfrac{k_2}{k_1}w_1x_2 + \dfrac{k_3}{k_1}w_1x_3}{w_1 + \dfrac{k_2}{k_1}w_1 + \dfrac{k_3}{k_1}w_1}$$

Simplifying,

$$\underline{x} = \frac{k_1x_1 + k_2x_2 + k_3x_3}{k_1 + k_2 + k_3} \tag{541}$$

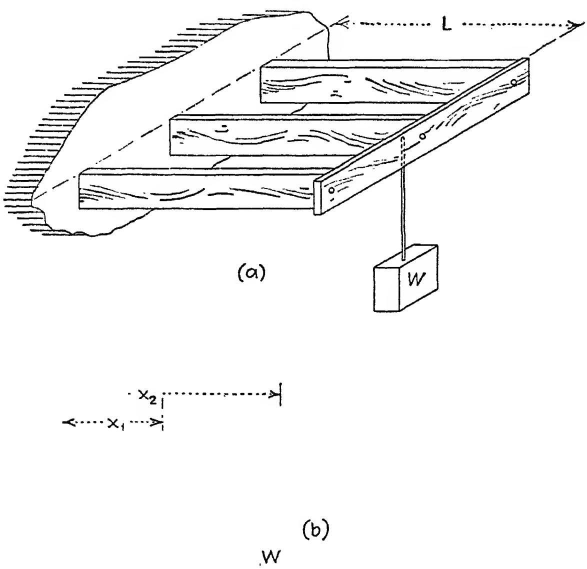

FIG. 158.—Elastic axis of three uniform spars.

Let us now consider the case of three spars, as illustrated in Fig. 158. Writing equations similar to (537), (538), and (539), we have

$$\Sigma F_y = 0$$

or

$$W = w_1 + w_2 + w_3 \tag{542}$$

Also,

$$\Sigma M_0 = 0$$

or

$$W\underline{x} = w_1x_1 + w_2x_2 + w_3x_3 \tag{543}$$

and

$$y_1 = \frac{w_1L^3}{3EI_1} = y_2 = \frac{w_2L^3}{3EI_2} = y_3 = \frac{w_3L^3}{3EI_3} \tag{544}$$

Solving these equations, we find

$$\underline{x} = \frac{I_1x_1 + I_2x_2 + I_3x_3}{I_1 + I_2 + I_3} \tag{545}$$

For any number of spars we may write

$$\underline{x} = \frac{I_1x_1 + I_2x_2 + \cdots I_nx_n}{I_1 + I_2 + \cdots I_n} \tag{546}$$

We note here that this equation locates the elastic axis at the end of the beams only, and for a concentrated load only. If the elastic curves of deflection of the beams are the same, the elastic

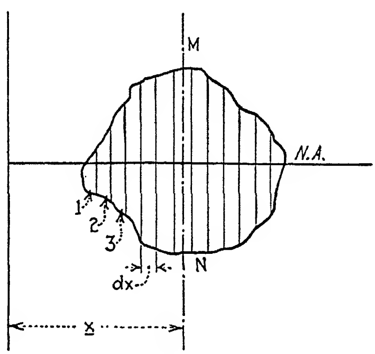

FIG. 159.—Location of elastic axis.

axis is a straight line. This implies that there exists a constant ratio at any section of the wing between I_1, I_2, and I_3. While the elastic axis of a cantilever wing, in most cases, is not a straight line, it appears that little error will be introduced by drawing a straight line as a mean curve through points obtained by successive application of equation (546). Further investigation is left for the student.

244. Elastic Axis, More General Equation.—It may be found desirable to use a more general equation than equation (546). Let us consider the subject further.

In Fig. 159, let MN represent the cross section of a beam of length L. We divide the beam into strips as 1, 2, 3, etc., of width dx. Thus, as in equation (546), we have

$$\underline{x} = \frac{I_1x_1 + I_2x_2 +}{I_1 + I_2 + \cdot \cdot \cdot} \tag{547}$$

We note that I is a function of x, as, for example, in Fig. 160,

$$I = \frac{y^3dx}{12}$$

and, since $y = kx$
we have

$$\frac{k^3x^3dx}{12} \tag{548}$$

Equation (547) becomes, therefore,

$$\underline{x} = \frac{\int Ixdx}{\int Idx} \tag{549}$$

in which I must be expressed in terms of x. If we take as a special case the value of I in equation (548),

$$\underline{x} = \frac{\displaystyle\int_0^d \frac{k^3x^4}{12}dx}{\displaystyle\int_0 \frac{k^3x^3}{12}dx} = \frac{4d^5}{5d^4} = \frac{4}{5}d \tag{550}$$

245. Elastic Axis and Shear.—The determination of the elastic axis in the few preceding paragraphs has been from the standpoint of bending only. While, in general, such determination is sufficient, it may be found that the shear in a particular type of wing is great enough to render such results too inaccurate. Let us consider the shear. With reference to Fig. 161, the modulus of elasticity in shear is

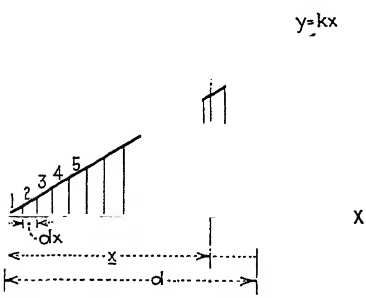

FIG. 160.—Elastic axis of a solid beam.

$$E_s = \frac{V}{A}\frac{dx}{dy} \tag{551}$$

or

$$y_s = \int_0^L \frac{V}{AE_s} dx \tag{552}$$

in which V is the shear at the section. For a concentrated load w at the end of the beam, V equals w so that

$$y_s = \frac{wL}{E_s A} \tag{553}$$

We note therefore that, in order to include the shear deflection, equation (544) would be written

$$y_1 = \frac{w_1 L^3}{3EI_1} + \frac{w_1 L}{E_s A_1} = \frac{w_2 L^3}{3EI_2} + \frac{w_2 L}{E_s A_2} - \frac{w_3 L^3}{3EI_3} + \frac{w_3 L}{E_s A_3} \tag{554}$$

It is now only necessary to solve equations (554), (542), and (543) for the value of x.

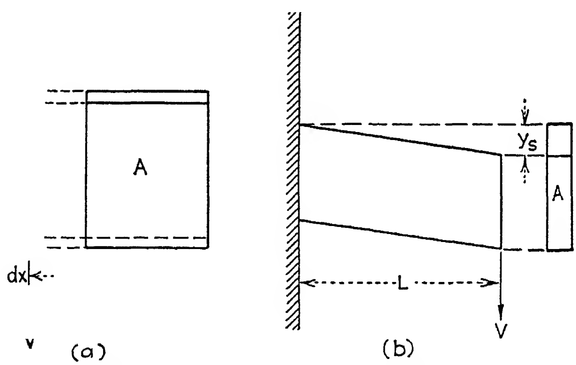

Fig. 161.—Elements of shear in a beam.

246. Trussed Spars.—The location of the elastic axis in a cantilever wing supported by trussed spars may be determined approximately by equation (546). However, as a large portion of the deflection is due to the deformation in vertical and diagonal bracing members, and since it is assumed in the development of equation (546) that the deflection is due solely to the extension and compression of outer fibers, as in the flange members of the spars, the result may not be sufficiently accurate in some cases. The following procedure may prove more desirable from the standpoint of accuracy: It will be noted that the general procedure is to determine w_3 and w_2 in terms of w_1 for equal deflec-

tions. Let us now first find the deflection of each truss for a load of, say, 1 lb., 100 lb., or 1,000 lb.—call this P. For example, suppose these deflections are found to be y_1, y_2, and y_3, and that y_2 is $2y_1$ and y_3 is $3y_1$. It is therefore apparent that for equal deflections $w_2 = \frac{1}{2}w_1$ and $w_3 = \frac{1}{3}w_1$. Thus in general, if $y_2 = k_2y_1$, $y_3 = k_3y_1$, and $y_n = k_ny_1$, we have

$$w_2 = \frac{w_1}{k_2}, \qquad w_3 \quad \frac{w_1}{k_3}, \qquad \text{and} \qquad w_n = \frac{w_1}{k_n} \tag{555}$$

Substituting this value in equation (543), we have

$$\underline{x} = \frac{w_1x_1 + \frac{w_1}{k_2}x_2 + \frac{w_1}{k_3}x_2 + }{w_1 + \frac{w_1}{k_2} + \frac{w_1}{k_3} + \cdots}$$

$$\frac{x_1 + \frac{x_2}{k_2} + \frac{x_3}{k_3} \cdots}{1 + \frac{1}{k_2} + \frac{1}{k_3} + \cdots}, \tag{556}$$

The deflections y_1, y_2, y_3, etc., may be found by any method, for example, that illustrated in Fig. 63. Several values of $\underline{x}$ along the semi-span may be computed and the average value of $\underline{x}$ taken as the required value. In general, the value near the tip of the wing is misleading, as one or more of the spars may radically change sections in that neighborhood. It is reasonable in computing the deflections at various points along the semi-span to neglect the portion of the spar from the station under consideration to the tip of the wing.

247. Requirements for Design.—An analysis of a cantilever wing for design purpose requires that the following applied loading characteristics be available:

1. Wing loading as a function of the chord. This is for rib design only.
2. Wing loading as a function of the semi-span. Specifically this is a curve showing the load per inch of semi-span of the wing as a function of the distance in inches from the tip or from the root section.
3. Shear at all sections of the semi-span of the wing as a function of the semi-span.
4. Bending moment at all sections of the semi-span of the wing as a function of the semi-span.
5. Torsion in the wing at all sections of the semi-span as a function of the semi-span.

248. Distribution of Load Between Wing Spars.—If the wing twists under load, the distribution of loading on the wing spars is greatly affected. This distribution is therefore a function of the torsional rigidity. If the wing is perfectly rigid in torsion, or if the load is applied so that torsion is not induced in the wing, the wing bends as a unit; that is, the deflection curves of the several spars are parallel. Under such pure bending, with identical deflection curves, the wing as a whole may be considered as a

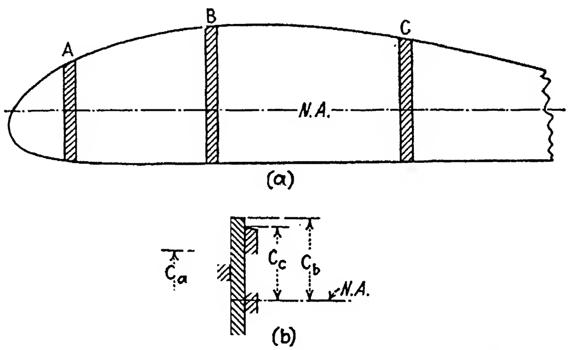

FIG. 162.—Distribution of stress for pure bending.

beam. The several spars may be considered as one composite spar for the purpose of calculation of stresses and division of loads. As an example, consider Fig. 162. The spars A, B, and C have been brought together in b to form a composite beam. With the neutral axis computed in the usual manner for a composite beam, we have [see equation (203)],

$$f = \frac{Mc}{I} \qquad \text{or} \qquad M = \frac{fI}{c} \tag{557}$$

in which $I = I_a + I_b + I_c$
so that

$$f = \frac{Mc}{(I_a + I_b + I_c)} \tag{558}$$

We note that (Fig. 162),

$$f_a{:}f_b = c_a{:}c_b, \qquad \text{and} \qquad f_c{:}f_b = c_c{:}c_b \tag{559}$$

Thus when f_b is the maximum allowable,

$$f_a = f_b\frac{c_a}{c_b} \qquad \text{and} \qquad f_c = f_b\frac{c_c}{c_b} \tag{560}$$

We note that the bending moment carried by each beam is

$$M_a = \frac{f_a I_a}{c_a}, \qquad M_b = \frac{f_b I_b}{c_b}, \qquad M_c = \frac{f_c I_c}{c_c} \tag{561}$$

We have

$$M = M_a + M_b + M_c \tag{562}$$

Equations (560), (561), and (562) enable us to solve for the portion of the bending moment carried by each beam. Thus, making substitutions from (560) into (561), we obtain

$$M_a = \frac{f_b I_a}{c_b}, \qquad M_b \quad \frac{f_b I_b}{c_b}, \qquad \text{and} \qquad M_c = \frac{f_b I_c}{c_b} \tag{563}$$

From which

$$f_b \qquad M_a c_b - M_b c_b - M_c c_b \tag{564}$$

Thus

$$M_a = M_b\frac{I_a}{I_b} \qquad \text{and} \qquad M_c = M_b\frac{I_c}{I_b} \tag{565}$$

Substituting these values of moments in equation (562), we have

$$M = M_b\frac{I_a}{I} + M_b + M_b\frac{I_c}{I} \tag{566}$$

From which

$$M_b = M\frac{}{I_a + I_b + I_c} \tag{567}$$

From symmetry, we write

$$M_c = M\frac{I_c}{I_a + I_b + I_c} \qquad \text{and} \qquad M_a = \frac{}{I_a + I_b + I_c} \tag{568}$$

Since the bending moment on a beam is proportional to the magnitude of the loading, assuming similar loading curves for the three beams, we may write for any section, equations similar to (567) and (568), for the loading on each beam.

We note particularly that the design stresses are obtained from equation (558), in which c may be c_a, c_b, or c_c, and that *we have assumed no torsion.*

249. Cantilever Wing as a Beam.—It has been noted that, under the assumption of no torsion in a wing and no relative motion between the spars, we may consider the entire wing as a beam. This has been shown, experimentally, for a stressed-skin wing. However, for any cantilever wing with ribs of sufficient

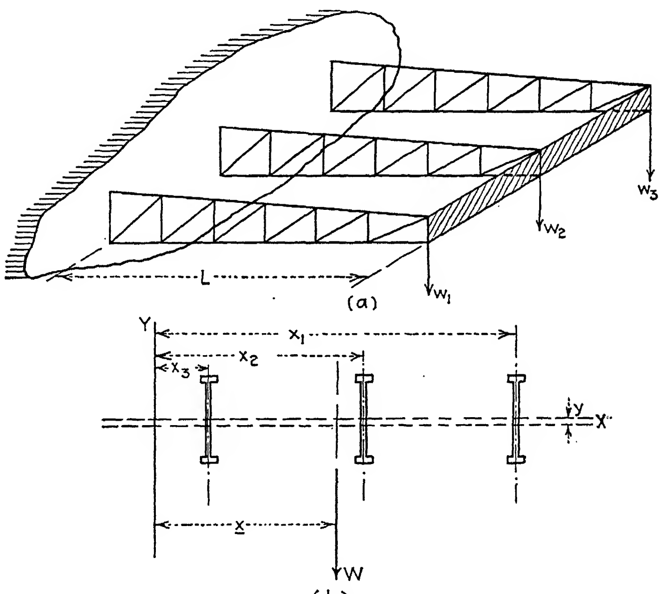

FIG. 163.—Bending, without torsion, in cantilever-trussed spar wing.

size and rigidity to cause the spars to retain the same deflection curve, the stresses caused by the bending component of the wing loading may be calculated by the modulus-of-rupture formula [see equation (557)].

In fact, since stresses in trussed spars and other supporting members in wings are usually below the elastic limit of the material, being designed for the compressive stress and columnar action, this formula applies with a fair degree of accuracy. It will be noted in this connection that f, the fiber stress, is also the P/A value of the chord or flange members of the wing trussing. This P/A, allowable value, is determined from Euler's or John-

son's column formulas, or by other methods appropriate to the conditions. We have thus

$$\frac{F}{A} \quad \frac{Mc}{I} \tag{569}$$

Let us consider a simple example to fix ideas. In Fig. 164 we have represented a wing of four trussed spars 1, 2, 3, and 4. Let us assume that the allowable columnar stress value P/A for each

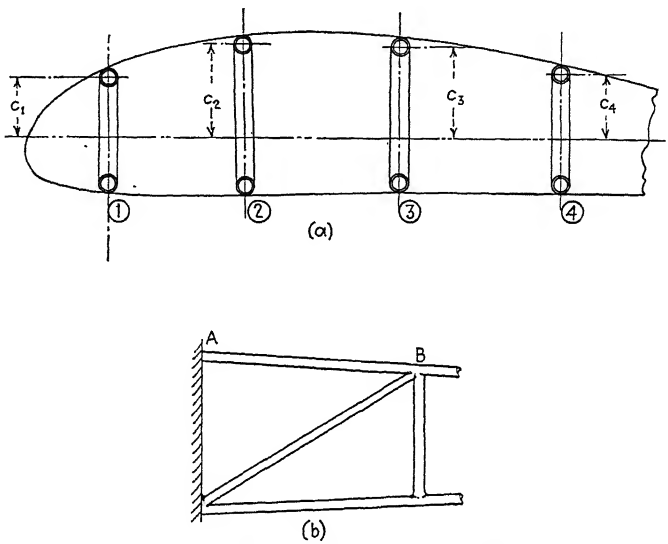

FIG. 164.—Wing with four trussed spars.

spar has been determined for the section as for AB in Fig. 164*b*, by appropriate column formulas or by experiment. Let this value, for example, be 30,000 lb. per square inch for spar 2. Thus

$$30{,}000 = \frac{Mc_2}{(I_1 + I_2 + I_3 + I_4)} \tag{570}$$

from which the allowable bending moment may be found; or, if the bending moment is known, the P/A value required may be found.

We note that, if c_1 is, say, one-half of c_2, the P/A load in the chord or flange members of spar 1 will be only half that of spar 2, or 15,000 lb. per square inch. Since the failure of spar 2 would in general mean the failure of the wing, spar 1 has considerable excess strength if its P/A value is also 30,000 lb. per square inch.

250. Combination of Light and Heavy Construction.—If a light spar is inserted in a wing between two heavy spars as a former for the wing ribs, the stress in the flanges of the light spar may be computed by formula (569). For example, if a light spar is inserted between spars 2 and 3 in Fig. 164a, with c approximately equal to c_2, the stress in the light spar must be approximately equal to the stress in spar 2. Since it is of light material, the P/A values obtainable in the light spar will be of the order, say, of half the stress of the large spar. The wing will therefore

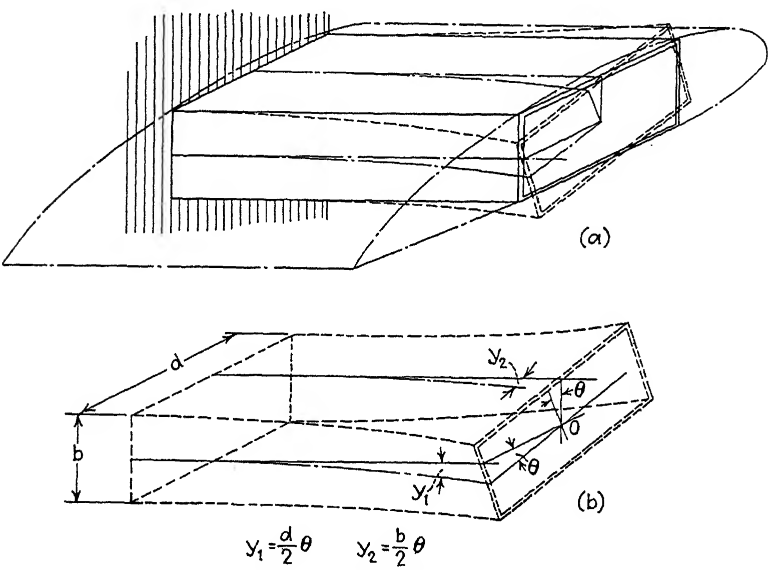

FIG. 165.—Twisted box spar.

begin to fail with a very much smaller load than may be ultimately obtained on the wing. This is the usual condition when light and heavy sections of materials are used for parallel services in a statically indeterminate structure. It is this effect which permits considerable wrinkling in the metal skin of a wing before danger of failure of the structure occurs.

251. Torsion in a Box-wing Beam.—A stressed-skin wing usually has for its supporting member a box beam. This box is usually the central section of the wing with the leading and trailing edges of the wing removable. An approximation of such a wing is shown in Fig. 165a. When a wing of this construction is twisted by a torsional couple Wx (Fig. 156) through an angle θ,

as noted in Fig. 165*b*, the couple is resisted by a shearing force as calculated from equation (431), and by the bending forces induced in the four sides by the deflection y_1 and y_2. Figure 166 shows the resulting interaction between the narrow and broad sides of the beam caused by this bending action in torsion. In *a* consider a Δx length BA. In *b*, with the broad side unrestrained by the action of the narrow side, the Δx length is com-

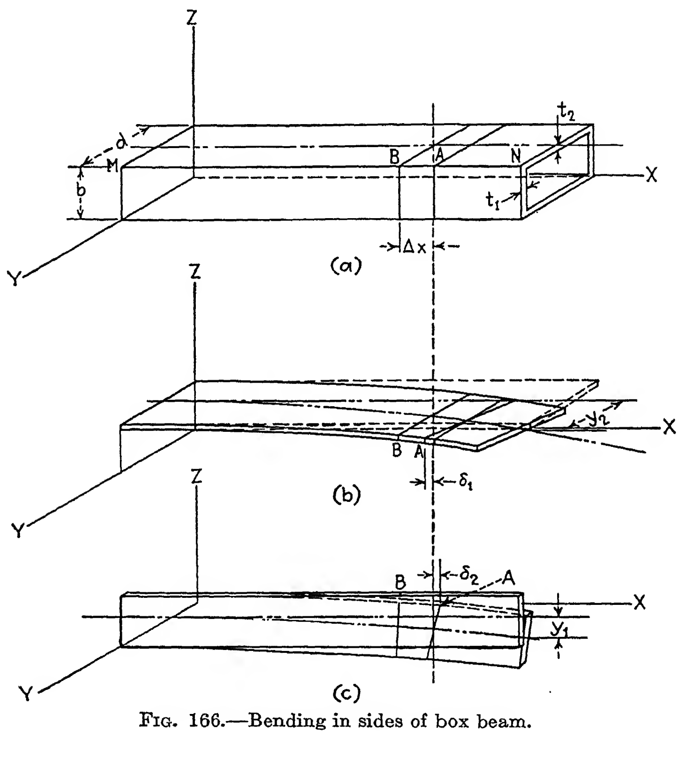

FIG. 166.—Bending in sides of box beam.

pressed a distance δ_1 by the bending action. In *c*, with the narrow side unrestrained by the broad side, the Δx length is extended a distance δ_2 by the bending action. This implies that the point A, considered as a part of the broad side, has moved to the left a distance of δ_1, and, considered as a part of the narrow side, has moved to the right a distance δ_2. In the first case a compressive stress in the fiber results, and in the second case a tensile stress results. It is apparent therefore that in the box beam *a* the shear restraint at the edge, preventing the point A

from moving freely to the left or right, tends to neutralize the tensile and compressive deformation, and hence to neutralize the tensile and compressive bending stresses.

As a simple example of such neutralization of tensile and compressive stresses, consider the torsion of a square box beam $b = d$ with $t_1 = t_2$. From the symmetry of the structure it is obvious that the point A will move neither to the left nor to the right; hence no tensile or compressive stress is developed at that point. This condition results, according to the theoretical analysis, when

$$t_1 b = t_2 d \qquad (571)$$

When $t_1 b$ is greater than $t_2 d$, the point A moves to the right, and hence a tensile stress is developed.

In stressed-skin wings, $t_2 d$ (of the top and bottom flanges) is greater than $t_1 b$ (of the webs); therefore a twist of the wing decreases the bending stresses in the webs (considered as spars).

In general, it appears that in stressed-skin wings the torsion in the wing does not increase the bending stresses (of tension and compression).

The shears induced by torsion and bending, however, are added to obtain the design shear.

252. Design of a Wing Beam.—Whether the wing itself is a beam, or whether the wing is supported by beams, the general principles in design are approximately the same. Assuming that we have obtained loading, shear, moment, and torsion curves for the wing, we are ready to proceed with the structural design of the beams. We must first decide on the specific type of construction with respect to:

1. Trussed beams.
2. Single-spar box beam; stressed skin
3. Web beams.
4. Material—as steel, duralumin.
5. Attachment of ribs with ease and efficiency.
6. Ease in manufacture; riveting, welding, forming.
7. Availability of standard sections.

The exact type of construction will, of course, be the result of careful study by the engineering force for the specific job. The final design will be the result of successive approximations or trials, much study, ingenuity, and a large number of experiments.

253. Chord Members.—Let us consider, for example, the chord or flange member of a trussed spar. Assume riveted construc-

tion. Assume the spar to be of composite construction as shown in Fig. 167, for example, with section *a* and *b*. Now the problem before us is to design the spar for minimum weight at each section along the spar. Since material cannot be purchased in tapered form, it is necessary to rivet successively thinner sections in place as we move outward on the spar toward the tip of the wing. It is obvious that it would not be desirable to make too many riveted joints because of the expense and additional weight, yet the spar must not be of the same heavy construction throughout its length. There is therefore a "happy medium" or compromise in the design of each spar concerning the location and number of joints.

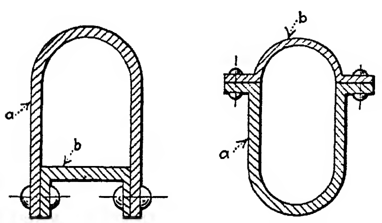

FIG. 167.—Types of tubular flange members.

254. Type of Joints.—The type of riveted joints depends upon the particular section under consideration. In some cases it is more desirable and less expensive to telescope the thinner section into the thicker section and rivet as a lap joint. A butt joint with a wrapping of the plate entirely around the spar section may be desirable if construction problems permit.

If a member is composed of three, four, or more, strips as *a* and *b* of Fig. 167, each strip may be successively decreased in size. This is the most desirable method.

255. Strength of the Section.—In general the chord or flange member is designed for compression. This is true because the allowable fiber stress in compression (as a strut) is generally less than half the allowable stress in tension. Each chord member, the top or the bottom member, will, of course, because of the inverted flight condition, be subjected to both types of loads. This condition makes the riveting problem simpler, since the design stresses of the material are rarely above the elastic limit.

In general no data will be available for computing the strength of the chosen chord section in column action. It may be necessary to perform a series of experiments on the chosen section to obtain a column curve of P/A and a function of L/ρ as is usually done, or to obtain simply a curve of P as a function of the particular section.

For preliminary design the following approximations may be found useful:

1. If the section is composed of approximately ⅛-in. plates, curved section, with width or depth of approximately 2 or 3 in., an allowable stress for duralumin of 25,000 to 30,000 lb. per square inch may be chosen.
2. If the section, as in *a*, is composed of approximately $\frac{1}{16}$-in. plates, a stress of 15,000 to 20,000 lb. per square inch may be chosen.
3. If the section, as in *a*, is composed of approximately 0.030-in. plate, a stress of 5,000 to 10,000 lb. per square inch may be chosen.

These stresses, of course, may be used only for very rough estimates of the general size and shape of the spar. The actual allowable stresses may be determined only by experiment.

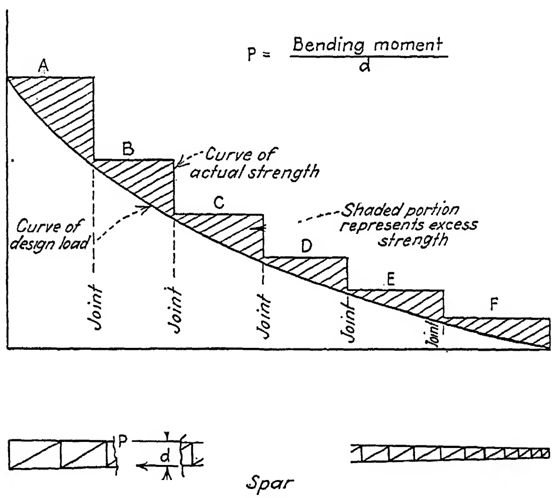

FIG. 168.—Preliminary design of a cantilever spar.

256. Preliminary Estimates.—To afford a visualization of the preliminary estimate in the design of the chord or flange member, a chart similar to that of Fig. 168 is convenient. The abscissa of this chart is the spar; the ordinate is the load in the chord member. This load is obtained by dividing the mean depth of the spar by the bending moment at the section.

If the material were of the same thickness throughout, its strength would be indicated by a continuation of the line *A*. However, all strength above the "design-load" line is excess strength, and the extra material is excess material. The actual strength, however, must always be above the design-load line. We are therefore able to visualize the best location of joints, the number of joints, the thickness of material, and other preliminary design items.

257. Vertical and Diagonal Members, Riveting.—Obviously a chart similar to that of Fig. 168 may be constructed for each design condition for vertical and diagonal members, and for riveting. A dozen or more such charts will therefore be necessary for the complete preliminary design.

It may be necessary, if the design is to be close, to build up sections of the beam for testing with the design loads, to obtain precise data on the allowable fiber stresses.

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1. Brown, C. G., and Captain C. F. Greene: Static Test and Stress Distribution Studies of the Material Division 55-foot Cantilever All-metal Wing, *Air Corps Information Circular* 663, 1932.

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CHAPTER XV

WING FLUTTER AND OTHER STRUCTURAL VIBRATIONS

258. Contributing Causes.—The conditions in modern airplane structures are ideal for the development of excessive vibrations. These basic conditions are:

1. An elastic structure.
2. Augmenting impulses such as contributed by the engines and air stream.

The combination of conditions may be expressed in a fundamental equation

$$M\frac{d^2y}{dt^2} + Ky = P_0 \sin \frac{2\pi}{T}t \tag{572}$$

in which

M = the mass of the vibrating structure.
K = the elasticity or spring constant of the structure.
$P_0 \sin \frac{2\pi}{T}t$ = the periodic augmenting force.
T = the period of the augmenting force.

To fix ideas, let us assume a weight suspended from a spring. In this case M is the mass of the weight, and K is the spring constant in pounds per foot of stretch. Now if the weight is vibrating freely (with no augmenting force), the equation is

$$M\frac{d^2y}{dt^2} + Ky = 0 \tag{573}$$

which is the equation of *simple harmonic motion,* the natural period of which is

$$T_0 = 2\pi\sqrt{\frac{M}{K}} \tag{574}$$

Now applying a periodic force, we express the conditions by equation (572).

259. Resonant Vibration.—The solution of equation (572) is

$$y = A \sin \sqrt{\frac{K}{M}}t + B \cos \sqrt{\frac{K}{M}}t + \frac{P_0/M}{\frac{K}{M} - \left(\frac{2\pi}{T}\right)^2} \sin \frac{2\pi}{T}t \qquad (575)$$

Now from equation (574) we note that

$$\frac{K}{M} = \left(\frac{2\pi}{T_0}\right)^2 \qquad (576)$$

Thus, with reference to the third term of equation (575), when $T_0 = T$, the denominator of the term becomes zero; hence y becomes infinite. Therefore, *when $T_0 = T$, that is, when the period of the augmenting force becomes equal to the natural period of vibration of the structure, the system is said to be in resonance.* In the prevention of dangerous structural vibration, the first principle is to avoid resonance. It will be observed [equation (574)] that the magnitude of P_0 affects ordinary vibration, but it does not affect resonant vibrations.

260. Transverse Vibration of Engine Mounts.—The frequency of impulsive forces of the engine, in general, is proportional to the r.p.m., and hence cannot be changed. It is easy, however, to fix the natural frequency (or period) of the structure, so that resonance will not occur. The frequency may be *raised* by making the structure more rigid. This may be accomplished by the use of more members, of heavier members, or of material with higher modulus of elasticity, or by an inherently rigid type of structure, such as monocoque. The natural frequency may be lowered by making the structure more flexible. Rubber or spring pads for mounting the engine are convenient for this purpose.

In calculating the natural frequency, equation (574) is used. M is the mass of the engine—the weight in pounds divided by 32.2. y is measured at the center of gravity of the engine in feet. K is the spring constant of the engine mount, that is, the force which would be required to deflect the structure 1 ft. For example, calculate the deflection of the structure due to a weight of 1,000 lb. If this is, say, 0.1 ft., then K = 10,000 lb. per foot. The deflection y may be simply computed as noted in Par. 128.

261. Torsional Vibration of Engine Mount.—The equation of motion for this condition is (Fig. 169),

$$I\frac{d^2\theta}{dt^2} + 2Kr^2\theta = M_0 \sin \frac{2\pi}{T}t \tag{577}$$

In this I is the mass moment of inertia of the engine about its center of oscillation, which may be assumed the center of gravity, K is the spring constant of the mounting, and M_0 is the applied torque. The solution is

$$\theta = A \sin \qquad t + B \cos \sqrt{2Kr^2} + \frac{M_0/I}{2Kr^2 \quad \left(\frac{2\pi}{T}\right)^2} \sin \frac{2\pi}{T}t \tag{578}$$

The natural period of oscillation is

$$T_0 = 2\pi$$

Resonance occurs when $T_0 = T$.

The units are pounds, feet, and seconds.

262. Damping Vibrations.—The usual methods of eliminating vibrations are as follows:

1. Remove, if possible, the source of the augmenting force. In the case of the engine, this would imply perfect balancing.
2. Change the natural frequency of the structure so that resonance with augmenting forces is not probable.
3. Dissipate the energy of the vibrating system by means of dry friction or viscous friction. Fibrous packing for joints in the structure is available commercially. Such packing between the engine mount and the structure and between the fuselage and wing is advantageous.

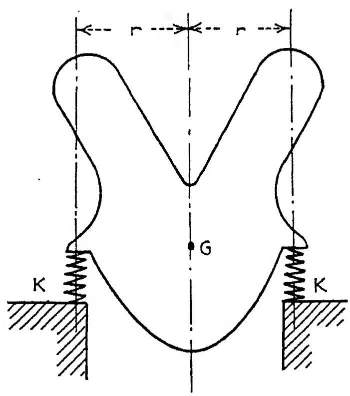

FIG. 169.—Flexible engine mount.

4. Set up an opposing vibrating system by means of a dynamical vibration damper (see Reference 6).

Methods (2) and (3) are most adaptable to aircraft.

263. Nature of Wing Flutter.—Wing flutter may occur in the overhanging tip of a braced wing or in a monoplane wing. The phenomenon is a vibration of the wing in approximately its own *natural frequency* and *mode.* The augmenting force is supplied

by the air stream, controlled by the action of the wing. The resulting amplitudes of vibration *build up* to alarming proportions, and generally the result is a destruction of the wings.

A wing has two principal modes of vibration. If the *elastic axis* coincides with the *gravity axis*, the modes are *pure bending* and *pure torsion*. If the elastic axis and the gravity axis do not coincide, the axes of vibration are inclined to the pure-bending

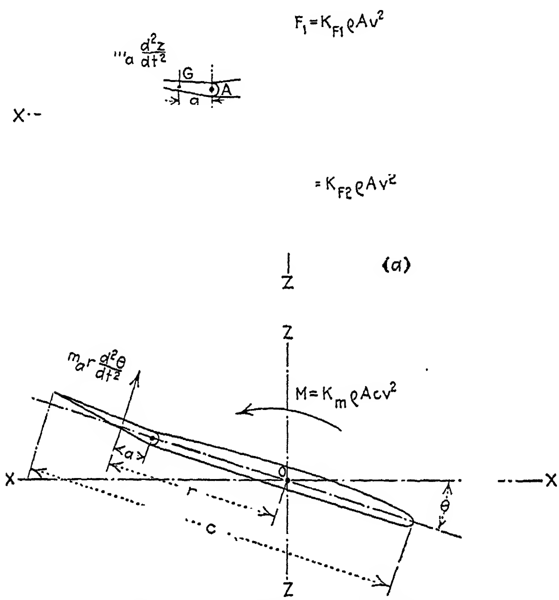

FIG. 170.—Modes of wing flutter.

and pure-torsional axes. For practical designs it is probably accurate enough to assume vibration in pure torsion and pure bending.

2 4. Wing Flutter in Bending.—In Fig. 170*a* we have represented the cross section of a cantilever monoplane wing near the tip of the wing. We assume, first, a free vibration of the wing in bending. Thus the "reversed effective force" acting on each mass particle is $m\frac{d^2z}{dt^2}$ acting away from the x-axis. Hence the moment tending to turn the aileron about its hinge is

$$= \left(m_a\frac{d^2z}{dt^2}\right)a \tag{579}$$

in which m_a is the mass of the aileron, assumed concentrated at the center of gravity G.

Now if the aileron is free, or if the elastic restraints (control wires) are flexible enough, the aileron is deflected as noted in the figure. If the wing is moving forward at a velocity V, the camber of the wing in its up-and-down position causes air impulses F_1 and F_2 respectively, where

$$F = K_F \rho A v^2 \tag{580}$$

in which the magnitude of K_F is unknown.

This augmenting impulse F causes the amplitude to build up to excessive values.

Experiments show that, when the natural frequency of the aileron about its hinge A is the same as the natural frequency of the wing in bending, this type of flutter is certain to occur. This is, in general, true also if the ailerons are free.

265. Wing Flutter in Torsion.—Figure 170*b* shows that the moment which tends to deflect the aileron is

$$T_a = \left(m_a r \frac{d^2\theta}{dt^2}\right)a \tag{581}$$

in which θ is the angular displacement of the wing. A restoring aerodynamic moment

$$M = K_m \rho A C v^2 \tag{582}$$

is induced which augments the oscillations. If the aileron is free, or if it has the same natural frequency in vibration about its hinge as that of the torsional oscillation of the wing, the amplitude will readily build up.

266. Aileron-wing Flutter.—It is readily obvious that the two types of flutter represented in Fig. 170 may occur simultaneously. In most cases this is probably what actually occurs. In this case, the point O, the center of oscillation, would be forward of the leading edge.

267. Recommendations for the Prevention of Wing Flutter.—In general, the structure of the wing should be as rigid as possible, though this alone will not preclude wing flutter. It is especially desirable that the wing be rigid in torsion. This is one of the most outstanding advantages of the stressed-skin wing—it is the most rigid in torsion of all types yet developed.

With reference to Fig. 170 and the discussion thereof, we may make the following recommendations relative to the ailerons:

1. The ailerons should be made as light as possible to decrease the deflecting moment [see equations (579) and (581)]. Some find it desirable in "all-metal" airplanes to cover the ailerons with fabric for lightness.
2. Make a in Fig. 170 zero or negative; that is, the hinge of the aileron should coincide with the center of gravity of the aileron, or be slightly to the rear of it.
3. The control rods or wires should be made as rigid as possible. From this standpoint, push rods or tubes are highly desirable. Long wires allow the ailerons considerable elastic movement with a low natural frequency, both of which are conducive to flutter.
4. Irreversibility of aileron controls prevents free oscillations of the ailerons, thus affording a damping tendency. Pilots, however, offer serious objections to this feature, as forces on the ailerons cannot be felt on the control wheel.
5. Frictional damping of the aileron has a decided effect in preventing aileron flutter, but it interferes with the smooth operation of the aileron.

The first three items of this list are the most effective in preventing wing flutter, and are under the control of the structural designer. The designer will find that item (4) is also quite effective if the aileron is made partially irreversible and the mechanism is installed immediately adjacent to the ailerons. The principle of the screw or inclined plane may be used effectively in this respect. The installation of the irreversible mechanism near the aileron makes the aileron quite rigid in torsion about its hinges.

268. Elevator Flutter.—Elevator flutter manifests itself as a torsional oscillation about the axis of the fuselage. Figure 170 may well represent the tip of a horizontal stabilizer with its elevator. In this case the forces and moments are of the same nature as those of the wings and ailerons.

Recommendations for the prevention of this type of flutter may be listed as follows:

1. Great torsional rigidity of the fuselage.
2. The elevators should be constructed on the same torsion tube. This tube between the elevators should be very rigid in torsion.
3. The five recommendations of the last paragraph for ailerons apply equally as well for elevators.

These rules will also apply for *rudders*.

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CHAPTER XVI

RIVETING IN AIRCRAFT CONSTRUCTION

269. Kinds of Riveted Joints.—Two general types of riveted joints in aircraft construction are (1) *lap joints* (Fig. 171), and (2) *butt joints* (Fig. 172).

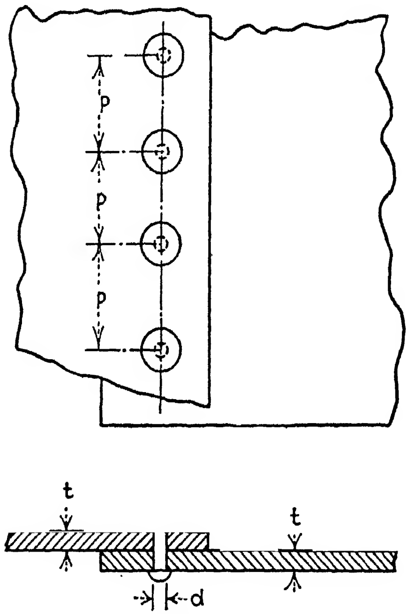


FIG. 171.—Lap joint.

It is seldom that two very thin sheets (less than 0.050 in. thick) are spliced except where a structural member backs up the joint as shown in Fig. 173. The designer is mostly concerned with the design of joints in strength members, gusset plates, etc., as shown in Figs. 174 and 175.

Either type of riveted joint may have one or more rows of rivets as shown in Figs. 176 and 177.

In the double-riveted lap joint (Fig. 178), the rivets in the second row may be placed directly behind the rivets in the first row, or they may be arranged zigzag as shown in the figure. The two rows must not be placed too closely together or the sheets

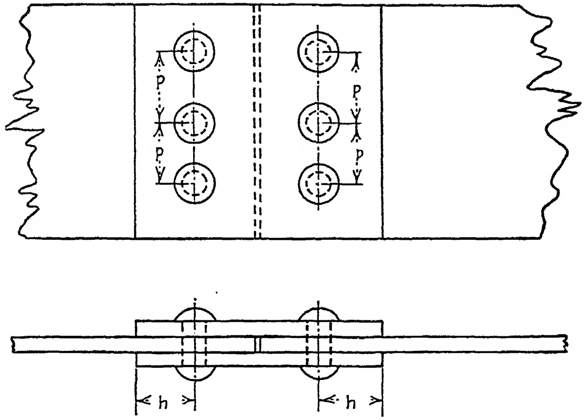


FIG. 172.—Butt joint.

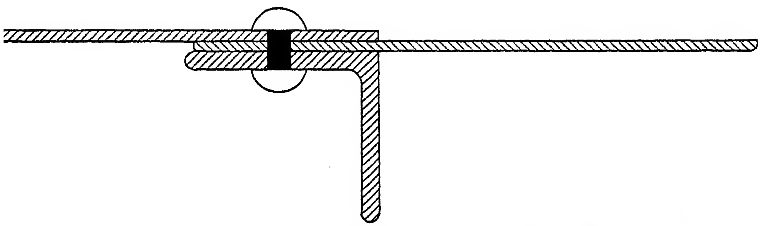


FIG. 173.—Riveted seam in thin material.

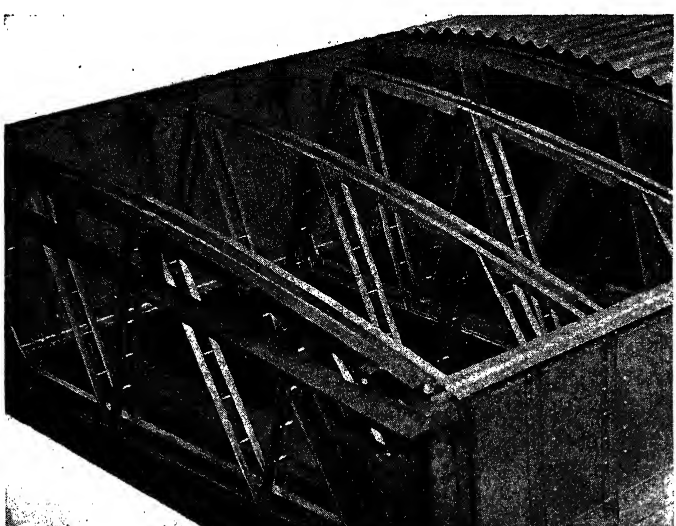


FIG. 174.—Riveted wing construction. (*Courtesy of Glenn L. Martin Company.*)

of thin metal may fail along the diagonal lines joining the rivets of the two rows. This is also true for all multiple-riveted joints.

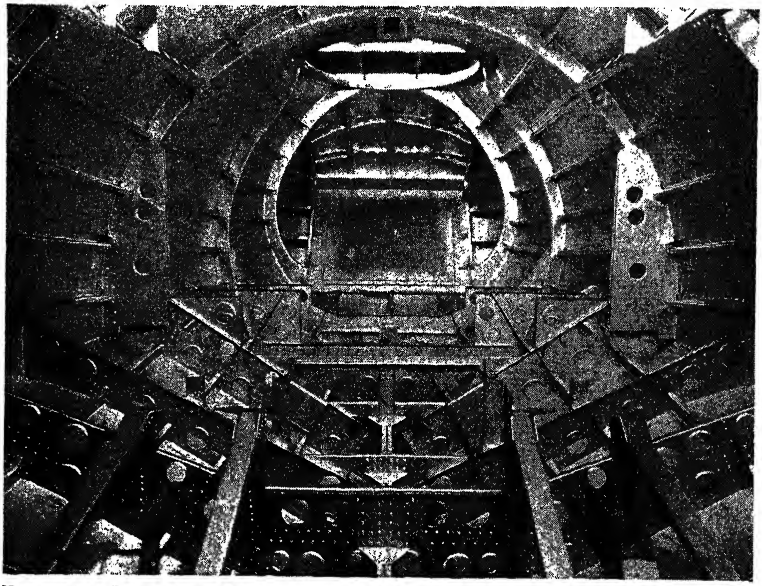


FIG. 175.—Riveted semi-monocoque construction of hull. (*Courtesy of Glenn L. Martin Company.*)

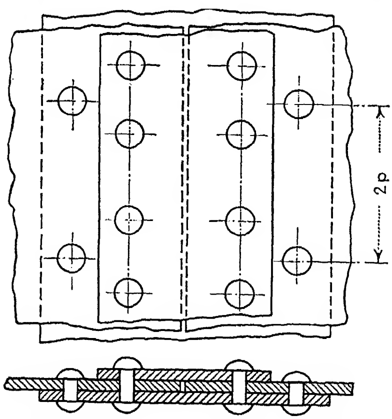


FIG. 176.—Multiple-riveted butt joint, with two rows of rivets.

The total distance a or b should be over 20 per cent greater than d. The rivets should be far enough away from the edge to prevent shearing out the plate. The pitch p depends upon the purpose

of the joint. In a gas tank the pitch is made quite small to prevent leakage. In a strength member the pitch, together with diameters of rivets, etc., should be calculated to give a joint of approximately the same strength in *tension, compression,* and *shear.*

270. General Rules for Design of Riveted Joints.—The designer should bear in mind, in the design of riveted joints, the following rules:

1. The joint should be designed so that the rivets are in shear. Tension in rivets should be avoided.

2. In general, the rivets should be regularly and equally spaced.

3. Where butt straps are used, as in the butt joints, the straps should *not* be of less thickness than the main plate.

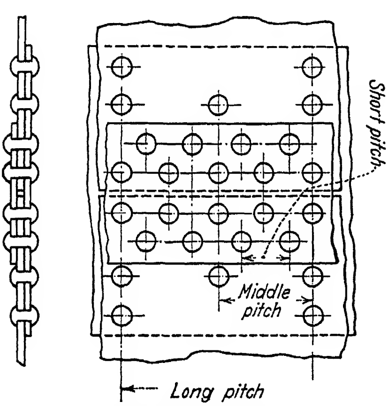


FIG. 177.—Multiple-riveted butt joint with four rows of rivets.

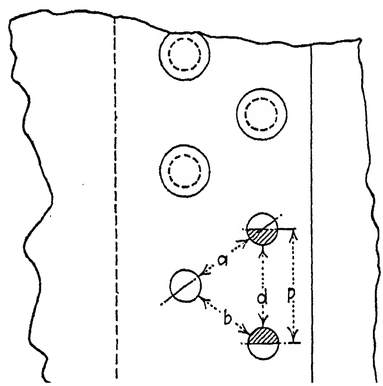


FIG. 178.—Spacing of rivets.

4. The thickness of the plates should regulate the size of the rivets (see Tables XX, XXI, XXII, and XXIII).

5. The rivet holes should have smooth and regular sides, without burrs or cracks. A drilled hole usually meets this requirement.

6. Rivets should be dipped into a good quality of bituminous paint or other preparation before inserting to prevent corrosion.

7. The sheets are bolted together with small bolts which fit the rivet holes before, and while, driving rivets.

8. In general, rivets should not be nearer to butts or edges of the plating, straps, or bars than a space equal to one and one-half times the diameter of rivet.

9. In general, in edge riveting, the space between any two consecutive rows of rivets should not be less than two rivet diameters.

271. Heat Treating Aluminum-alloy Rivets.—Rivets of 2S and 3S do not need heat treating before driving, although they do

"work-harden" in driving. This is a desirable feature to prevent permanent deformation under load and loosening as a result. It is imperative that 17S rivets be heat treated before driving as they are too hard normally and, if driven untreated, are very apt to split around the edge of the head. The hard rivet will also swell the hole in the sheet, causing it to wave between rivets.

The heat treatment of duralumin rivets is a comparatively simple operation, but the temperature must be properly controlled to obtain satisfactory results. Although there are several types of furnaces used for the treatment of rivets and small duralumin parts, the salt-tank type is the most widely used. This consists of a steel tank of sufficient capacity to handle the desired volume of work, with some means of heating (electricity, gas, etc.). The salt bath, which is a half-and-half mixture of pure sodium nitrate and potassium nitrate, is heated to from 940 to 960°F. in the heat-treatment process, and is maintained at this temperature through the use of a pyrometer. The rivets are left in the solution 10 to 20 minutes, depending upon the size of the rivets. After the rivets have been in the salt bath for the required period, they should be taken out and quenched in cold water and then washed off. Two tanks should be used for the quenching and washing operations.

The rivets should be driven within one-half hour after quenching; otherwise they will become excessively hard and re-treatment will be necessary.

After they have been quenched, the rivets are soft, but start aging immediately, obtaining their maximum hardness in about four days. This feature is very advantageous as it allows sufficient time for the job to be finished before the maximum hardness is developed. If, for any reason, it is desired to keep the rivets soft after heat treatment, they may be placed in an atmosphere below 32°F., as in containers surrounded by dry ice (solid carbon dioxide). In this way the rivets may be kept soft enough for driving for several hours.

272. Riveting of Stainless-steel Sheets.—Riveting of stainless steel is done either hot or cold. Small rivets may be drawn cold and set by comparatively few heavy blows. Hot rivets should be heated out of contact with the furnace flame or in electric resistance heaters, which use the rivet as the circuit between two electrodes of low electrical resistance, to a temperature of 2100°F.

These hot rivets must be set so that mechanical deformation is finished before they cool below 1750°F. Riveting is of doubtful economy in structural joints. The holes are readily torn. Stainless rivets harden rapidly and are difficult to handle.

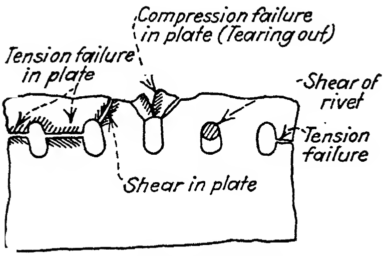
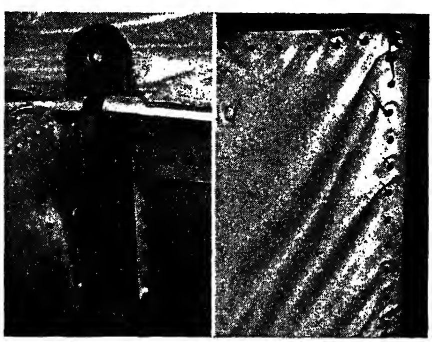


FIG. 179.—Types of failures in riveted joint.

273. Rivet Stress Calculations.—Figures 179 and 180 show the three types of failures in a rivited joint, in tension, in compression, and in shear. Figure 181 shows a free-body diagram of the types of stress in a riveted joint to be provided for in design. These may be listed as follows:

1. Shear in rivet.
2. Shear in plate.
3. Compression in plate.
4. Compression in rivet.
5. Tension in plate.
6. Tension in rivet (if designed for tension; this, in general is poor practice).



(a) (b)

FIG. 180.—Types of failures in riveted joints. (a) Tearing and shear of sheet, (b) tension and crushing of sheet.

In stress calculations it is assumed that a group of rivets in a joint will carry their full shear-stress value. If the joint is not properly designed, this assumption does not hold. The proper consideration must be given to the elasticity of the plates or straps. The stretch or strain in a strap is proportional to the stress; hence the first row of rivets is subjected to a greater shear

stress than are the remaining rivets. In a compact joint this condition does not affect the strength appreciably.

Care must also be exercised in grouping the rivets so that the centroid of the rivet areas coincides with the intersection of the neutral axes of the members connected; otherwise, because of the flexibility of the members, undue stress will be thrown upon some of the rivets by the bending moment developed by eccentricity. Experiments show that an error in design of this nature produces considerable loss in the strength of the joint.

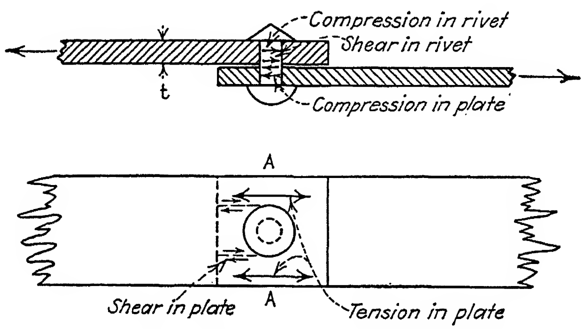


FIG. 181.—Types of stresses in riveted joint.

274. Joints of Equal Strength in Shear, Tension, and Compression.—If we designate s as the unit shearing strength of the rivets, f_c the unit crushing or compressive strength of the plate, f_t the unit tensile strength of the plate, t the thickness of the plate, d the diameter of the rivets, and p the pitch of the rivets, we have for a single-row riveted lap joint, by equating the compressive strength to the tensile strength of the plate,

$$f_c td = f_t t(p - d) \tag{583}$$

or

$$p = \frac{f_c + f_t}{} \tag{584}$$

If we define the efficiency as

$$e = p - \tag{585}$$

we have, by substituting equation (584) in equation (585),

$$e = \frac{f_c}{f_c + f_t} \tag{586}$$

If we equate the shearing strength and the compressive strength, we have

$$f_ctd = s\pi d^2 \tag{587}$$

or

$$d = \frac{4f_ct}{\pi s} \tag{588}$$

Thus, if the values of f_s, s, and t are known, the diameter and pitch of the rivets may be determined from equations (584) and (588).

For a single-riveted butt joint the relation between compression of plate and tension in the plate section is the same as in a single-riveted lap joint; hence equation (584) holds for this case. Since each rivet is in double shear, equation (587) becomes

$$f_ctd = 2s\pi d^2 \tag{589}$$

from which

$$= \frac{2f_ct}{\pi s} \tag{590}$$

In a double-riveted lap joint two rivets are in compression in unit width, so that

$$2f_ctd = f_tt(p - d) \tag{591}$$

Thus

$$p = \frac{2f_c + f_t}{f_t}d \tag{592}$$

The relation between shear and compression is the same as in a single-riveted lap joint so that equation (588) holds.

275. Eccentric Loads on Riveted Joints.—The following procedure should be followed when calculating rivet and bolt stresses in eccentrically loaded joints:

1. *Calculate the centroid of the rivet group.*—This is done in the same way as finding centroids of composite sections except that the rivet values are used instead of areas. Either the *shear* or *bearing* values should be used, depending on which is the smaller.

$$= \frac{V_1d_1 + V_2d_2 + V_3d_3 + \cdots}{V_1 + V_2 +} \tag{593}$$

where V = bearing or shear value of rivet, and d = distance of rivet from reference axis.

2. *Calculate the moment of the applied load about the centroid of the rivet group.*

3. *Calculate the moment of inertia of the rivet group about the centroid.*—This is done by taking the sum of the product of the rivet value and the square of the distance to the centroid.

$$I = \Sigma V_1 r_1^2 + V_2 r_2^2 + \cdots \tag{594}$$

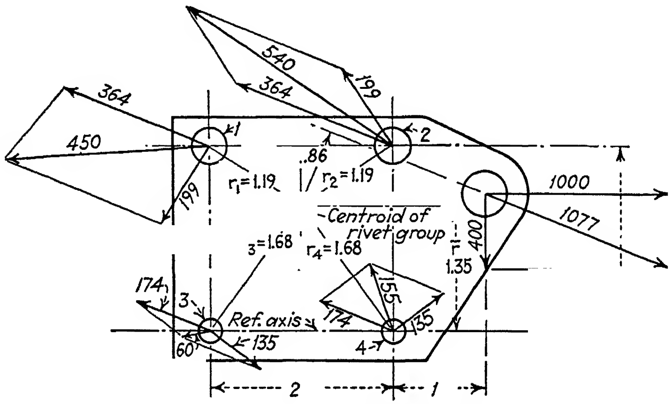


FIG. 182.—Centroid rivet group.

4. *Calculate the resultant load on each rivet due to the moment.*

$$\text{Load on No. 1 rivet} = \frac{M(V_1 r_1)}{\Sigma V r^2} \tag{595}$$

The direction of this load on each rivet is perpendicular to a line passing through the rivet and the centroid of the section.

5. *Calculate the load on each rivet due to the direct load.*—This is equal to the resultant applied load times the value of the rivet in question divided by the sum of all the rivet values.

$$\text{Direct load on rivet No. 1} = \text{resultant load} \times \frac{V_1}{\Sigma V}$$

6. *From inspection determine which rivets are most highly loaded, and find the resultant load on these rivets; this is the vector sum of the load due to moment and the load due to direct load.*

The following example is taken to show how the method is applied to a typical joint (see Fig. 182). The plate is aluminum alloy of 0.064 = in. thickness. The two upper aluminum-alloy rivets are $\frac{3}{16}$ in. in diameter and the two lower ones $\frac{1}{8}$ in. in diameter.

Bearing value of $\frac{3}{16}$-in. rivets on 0.064-in. plate = 900 lb.
Shear value of $\frac{1}{8}$-in. rivets = 431 lb. (assumed data).

It should be noted that in figuring the rivet values the critical one should be used. For example the $\frac{1}{8}$-in. rivet is weaker in shear than in bearing in this case.

It is generally more convenient to arrange the calculations in tabular form as shown below:

Tabulated Results

Rivet number (1)	Critical value, V (2)	d* (3)	Vd (4)	r (5)	Vr^2 (6)	Vr (7)	Moment load (8)	Direct load (9)
1	900	2.0	1,800	1.19	1,274	1,071	199	364
2	900	2.0	1,800	1.19	1,274	1,071	199	364
3	431	0	0	1.68	1,216	724	135	174
4	431	0	0	1.68	1,216	724	135	174

* The reference axis is taken to be the horizontal line through the two lower rivets.
$\Sigma V = 2662 \quad \Sigma Vd = 3600 \quad \Sigma Vr^2 = 4980$

$$\bar{r} = {}^{3600}\!/_{2662} = 1.35 \text{ in.}$$

M = applied moment = 1077 × 0.86 = 926 in.-lb.

The loads in column (8) were determined by multiplying the applied moment by the corresponding value in column (7) and dividing by the sum total of column (6). Thus

$$\text{"moment load"} = \frac{M(Vr)}{\Sigma Vr^2}$$

The loads in column (9) were determined by multiplying the resultant applied load by the corresponding rivet value given in column (2) and dividing by the sum total of the rivet values. Thus "direct load" $= P\dfrac{V}{\Sigma V}$

The vectorial sums of the rivet loads are shown in Fig. 182.

276. Strength of Plate Versus Strength of Rivet.—We should not use $\frac{1}{8}$-in. rivets to join $\frac{1}{4}$-in. plates, nor should we use $\frac{1}{4}$-in. rivets to join 0.020-in. plate. In the first case the rivet would shear off at a low-plate stress, and in the second case the plate would tear out at a low-rivet stress. It is probably desirable to design the joint so that the rivets will fail in shear before the plate fails by crushing, tearing, or shearing. Tables XX, XXI,

TABLE XX.—BEARING STRENGTH OF ALUMINUM-ALLOY SHEET—17S (SPEC. QQ-A-353)[1]

Diameter of rivet or pin, in.	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	7/16	1/2	9/16	5/8	3/4	7/8	1
Single shear	92	207	368	575	828	1,472	2,300	3,315	4,509	5,889	7,455	9,204	13,254	18,039	23,562
Sheet, thickness, in.	Bearing strength of sheet, lb.														
0.012	56	84	112	140	168	224	280	336	392	448	504	560	672	784	896
0.016	**75**	112	150	187	225	300	374	450	525	600	674	750	900	1,050	1,200
0.020	94	141	187	234	281	375	469	562	656	750	844	937	1,125	1,312	1,500
0.025	117	**176**	234	293	351	468	586	702	819	936	1,054	1,170	1,404	1,638	1,872
0.030	141	211	281	352	422	562	704	843	984	1,124	1,266	1,405	1,686	1,967	2,248
0.035	164	246	**328**	410	492	656	820	984	1,148	1,312	1,476	1,640	1,968	2,296	2,624
0.040	188	281	376	469	563	750	938	1,125	1,313	1,500	1,688	1,875	2,250	2,625	3,000
0.045	211	316	422	**527**	633	844	1,054	1,266	1,477	1,688	1,898	2,110	2,532	2,954	3,376
0.051	239	359	478	598	717	956	1,196	1,434	1,673	1,912	2,152	2,390	2,868	3,346	3,824
0.057	267	401	534	668	**801**	1,068	1,336	1,602	1,869	2,136	2,404	2,670	3,204	3,738	4,272
0.064	300	450	600	750	900	1,200	1,500	1,800	2,100	2,400	2,700	3,000	3,600	4,200	4,800
0.072	338	506	676	844	1,014	**1,352**	1,688	2,028	2,366	2,704	3,040	3,380	4,056	4,732	5,408
0.081	380	570	760	950	1,140	1,520	1,900	2,280	2,660	3,040	3,420	3,800	4,560	5,320	6,080
0.091	427	640	854	1,067	1,281	1,708	**2,134**	2,562	2,989	3,416	3,842	4,270	5,124	5,978	6,832
0.102	478	717	956	1,195	1,434	1,912	2,390	**2,868**	3,346	3,824	4,302	4,780	5,736	6,692	7,648
0.128	600	900	1,200	1,500	1,800	2,400	3,000	3,600	**4,200**	4,800	5,400	6,000	7,200	8,400	9,600
0.156	731	1,097	1,462	1,828	2,193	2,924	3,656	4,386	5,117	**5,848**	**6,580**	7,310	8,772	10,234	11,696
3/16	879	1,319	1,758	2,198	2,637	3,516	4,395	5,274	6,153	7,032	7,911	**8,790**	**10,548**	12,306	14,064
1/4	1,172	1,758	2,344	2,930	3,516	4,688	5,860	7,035	8,208	9,375	10,548	11,725	14,070	16,415	18,750

[1] Courtesy of U. S. Army Air Corps.
All values appearing below figures in bold face type in each column are greater than single shear.
Allowable bearing = 75,000 lb. per square inch. Allowable shear—17S (Spec. QQ-A-351) Rivet = 30,000 lb. per square inch.

TABLE XXI.—BEARING STRENGTH OF ALUMINUM-COATED ALUMINUM-ALLOY SHEET—17S ALCLAD (SPEC. 57-152-2)[1]

Diameter of rivet or pin, in.	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	7/16	1/2	9/16	5/8	3/4	7/8	1
Single shear strength, lb.	92	207	368	575	828	1,472	2,300	3,315	4,509	5,889	7,455	9,204	13,254	18,039	23,562
Sheet thickness, in.	Bearing strength of sheet, lb.														
0.012	51	76	102	127	153	204	255	306	357	408	459	510	612	714	816
0.016	68	102	136	170	204	272	340	408	476	544	612	680	816	952	1,088
0.020	**85**	127	170	212	255	340	425	510	595	680	765	850	1,020	1,190	1,360
0.025	106	159	212	265	318	424	530	636	742	848	954	1,060	1,272	1,484	1,696
0.030	128	**192**	256	320	384	512	640	768	896	1,024	1,152	1,280	1,536	1,792	2,048
0.035	149	223	298	372	447	596	745	894	1,043	1,192	1,341	1,490	1,788	2,086	2,384
0.040	170	255	**340**	425	510	680	850	1,020	1,190	1,360	1,530	1,700	2,040	2,380	2,720
0.045	191	286	382	477	573	764	955	1,146	1,337	1,528	1,719	1,910	2,292	2,674	3,056
0.051	217	325	434	**542**	651	868	1,085	1,302	1,519	1,736	1,853	2,170	2,604	3,038	3,472
0.057	242	363	484	605	726	968	1,210	1,452	1,694	1,936	2,178	2,420	2,904	3,388	3,872
0.064	272	408	544	680	**816**	1,088	1,360	1,632	1,904	2,176	2,448	2,720	3,264	3,808	4,352
0.072	306	459	612	765	918	1,224	1,530	1,836	2,142	2,448	2,754	3,060	3,672	4,284	4,896
0.081	344	516	688	860	1,032	**1,376**	1,720	2,064	2,408	2,752	3,096	3,440	4,128	4,816	5,504
0.091	387	580	774	967	1,161	1,548	1,935	2,322	2,709	3,096	3,483	3,870	4,644	5,418	6,192
0.102	434	651	868	1,085	1,302	1,736	**2,170**	2,604	3,038	3,472	3,906	4,340	5,208	6,076	6,944
0.128	544	816	1,088	1,360	1,632	2,176	2,720	**3,264**	**3,808**	4,352	4,896	5,440	6,528	7,616	8,704
0.156	663	994	1,326	1,657	1,989	2,652	3,315	3,978	4,641	**5,304**	5,967	6,630	7,956	9,282	10,608
3/16	797	1,195	1,594	1,992	2,391	3,188	3,985	4,782	5,579	6,375	**7,173**	**7,970**	9,564	11,158	12,750
1/4	1,063	1,594	2,125	2,657	3,189	4,250	5,315	6,378	7,441	8,500	9,565	10,630	12,756	14,882	17,000

[1] Courtesy of U. S. Army Air Corps.

All values appearing below figures in **bold face** type in each column are greater than single shear.

Allowable bearing = 68,000 lb. per square inch. Allowable shear—17S (Spec. QQ-A-351) Rivet = 30,000 lb. per square inch.

TABLE XXII.—BEARING STRENGTH OF ALUMINUM-ALLOY SHEET—24S (SPEC. 11066)[1]

Diameter of rivet or pin, in.	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	7/16	1/2	9/16	5/8	3/4	7/8	1
Single shear strength, lb.	107	242	430	671	966	1,717	2,684	3,868	5,261	6,871	8,697	10,738	15,463	21,046	27,489
Sheet thickness, in.	Bearing strength of sheet, lb.														
0.012	67	100	134	167	201	268	335	402	469	536	603	670	804	938	1,072
0.016	**90**	135	180	225	270	360	450	540	630	720	810	900	1,080	1,260	1,440
0.020	112	168	224	280	336	448	560	672	784	896	1,008	1,120	1,344	1,568	1,792
0.025	141	**211**	282	352	423	564	705	846	987	1,128	1,269	1,410	1,692	1,974	2,256
0.030	169	253	338	422	507	676	845	1,014	1,183	1,352	1,521	1,690	2,028	2,366	2,704
0.035	197	295	**394**	492	591	788	985	1,182	1,379	1,576	1,773	1,970	2,364	2,758	3,152
0.040	225	337	450	562	675	900	1,125	1,350	1,575	1,800	2,025	2,250	2,700	3,150	3,600
0.045	253	379	506	**632**	759	1,012	1,265	1,518	1,771	2,024	2,277	2,530	3,036	3,542	4,048
0.051	287	430	574	717	861	1,148	1,435	1,722	2,009	2,296	2,583	2,870	3,444	4,018	4,592
0.057	321	481	642	802	**963**	1,284	1,605	1,926	2,247	2,568	2,889	3,210	3,852	4,494	5,136
0.064	360	540	720	900	1,080	1,440	1,800	2,160	2,520	2,880	3,240	3,600	4,320	5,040	5,760
0.072	405	607	810	1,012	1,215	**1,620**	2,025	2,430	2,835	3,240	3,645	4,050	4,860	5,670	6,480
0.081	456	684	912	1,140	1,368	1,824	2,280	2,736	3,192	3,648	4,104	4,560	5,472	6,384	7,296
0.091	512	768	1,024	1,280	1,536	2,048	**2,560**	3,072	3,584	4,096	4,608	5,120	6,144	7,168	8,192
0.102	574	861	1,148	1,435	1,722	2,296	2,870	**3,444**	4,018	4,592	5,166	5,740	6,888	8,036	9,184
0.128	720	1,080	1,440	1,800	2,160	2,880	3,600	4,320	**5,040**	**5,460**	6,480	7,200	8,640	10,080	11,520
0.156	877	1,315	1,754	2,192	2,631	3,508	4,385	5,262	6,139	7,016	**7,893**	8,770	10,524	12,278	14,032
3/16	1,055	1,582	2,110	2,637	3,165	4,220	5,275	6,330	7,385	8,440	9,495	**10,550**	12,660	14,770	16,880
1/4	1,406	2,109	2,812	3,515	4,218	5,625	7,030	8,436	9,842	11,250	12,655	14,060	16,872	19,684	22,500

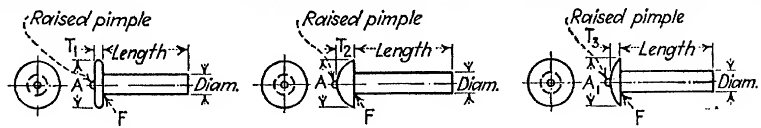
[1] Courtesy of U. S. Army Air Corps.
All values appearing below figures in bold face type in each column are greater than single shear.
Allowable bearing = 90,000 lb. per square inch. Allowable shear—24S (Spec. 11071) Rivet = 35,000 lb. per square inch.

TABLE XXIII.—BEARING STRENGTH OF ALUMINUM-COATED ALUMINUM-ALLOY SHEET—24S ALCLAD (SPEC. 11067)[1]

Diameter of rivet or pin, in.	1/16	3/32	1/8	5/32	3/16	1/4	5/16	3/8	7/16	1/2	9/16	5/8	3/4	7/8	1
Single shear strength, lb.	107	242	430	671	966	1,717	2,684	3,868	5,261	6,871	8,697	10,738	15,463	21,	27,489
Sheet thickness, in.	Bearing strength of sheet, lb.														
0.012	61	91	122	152	183	244	305	366	427	488	549	610	732	854	976
0.016	82	123	164	205	246	328	410	492	574	656	738	820	984	1,148	1,312
0.020	**102**	153	204	255	306	408	510	612	714	816	918	1,020	1,224	1,428	1,632
0.025	128	192	256	320	384	512	640	768	896	1,024	1,152	1,280	1,536	1,792	2,048
0.030	154	**231**	308	385	462	616	770	924	1,078	1,232	1,386	1,540	1,848	2,156	2,464
0.035	179	268	358	447	537	716	895	1,074	1,253	1,432	1,611	1,790	2,148	2,506	2,864
0.040	205	307	**410**	512	615	820	1,025	1,230	1,435	1,640	1,845	2,050	2,460	2,870	3,280
0.045	231	346	462	577	693	924	1,155	1,386	1,617	1,848	2,079	2,310	2,772	3,234	3,696
0.051	261	391	522	**652**	783	1,044	1,305	1,566	1,827	2,088	2,349	2,610	3,132	3,654	4,176
0.057	292	438	584	730	**876**	1,168	1,460	1,752	2,044	2,336	2,628	2,920	3,504	4,088	4,672
0.064	328	492	656	820	984	1,312	1,640	1,968	2,296	2,624	2,952	3,280	3,935	4,592	5,248
0.072	369	553	738	922	1,107	1,476	1,845	2,214	2,583	2,952	3,321	3,690	4,428	5,166	5,904
0.081	415	622	830	1,037	1,245	**1,660**	2,075	2,490	2,905	3,320	3,735	4,150	4,980	5,810	6,640
0.091	466	699	932	1,165	1,398	1,864	2,330	2,796	3,262	3,728	4,194	4,660	5,592	6,524	7,456
0.102	523	784	1,046	1,307	1,569	2,092	**2,615**	**3,138**	3,661	4,184	4,707	5,230	6,276	7,322	8,368
0.128	656	984	1,312	1,640	1,968	2,624	3,280	3,936	**4,592**	5,248	5,904	6,560	7,872	9,184	10,496
0.156	799	1,198	1,598	1,997	2,397	3,196	3,995	4,794	5,593	**6,392**	7,191	7,990	9,588	11,186	12,784
3/16	961	1,441	1,922	2,402	2,883	3,844	4,805	5,766	6,727	7,688	**8,649**	**9,610**	11,532	13,454	15,376
1/4	1,281	1,921	2,562	3,202	3,843	5,125	6,405	7,686	8,967	10;250	11,530	12,810	15,372	17,934	20,500

[1] Courtesy of U. S. Army Air Corps.
All values appearing below figures in bold face type in each column are greater than single shear.
Allowable bearing = 82,000 lb. per square inch. Allowable shear—24S (Spec. 11071) Rivet = 35,000 lb. per square inch.

TABLE XXIV.—S. A. E. STANDARD OF ALUMINUM-ALLOY RIVET DIMENSIONS[1]

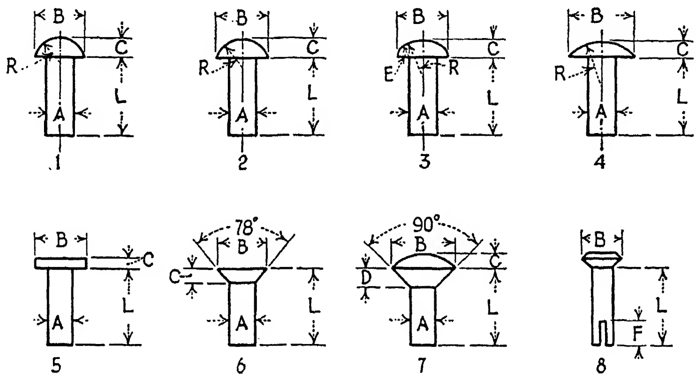


Body diameter in.	1/16 +0.003 −0.001	3/32 +0.003 −0.001	1/8 +0.0035 −0.001	5/32 +0.004 −0.001	3/16 +0.004 −0.001	1/4 +0.004 −0.001	5/16 +0.004 −0.001	3/8 +0.004 −0.001
A	1/8 ± 0.006	3/16 ± 0.009	1/4 ± 0.012	5/16 ± 0.016	3/8 ± 0.019	1/2 ± 0.025	5/8 ± 0.031	3/4 ± 0.037
A_1	5/32 ± 0.008	15/64 ± 0.012	5/16 ± 0.016	25/64 ± 0.020	15/32 ± 0.023	5/8 ± 0.031	25/32 ± 0.039	15/16 ± 0.047
T_1	0.025 ±0.005	0.038 ±0.005	0.050 ±0.005	0.062 ±0.005	0.075 ±0.005	0.100 ±0.005	0.125 ±0.005	0.150 ±0.007
T_2	0.047 ±0.005	0.070 ±0.005	0.094 ±0.005	0.117 ±0.005	0.141 ±0.007	0.188 ±0.009	0.234 ±0.012	0.281 ±0.014
T_3	1/32 ± 0.005	3/64 ± 0.005	1/16 ± 0.005	5/64 ± 0.005	3/32 ± 0.005	1/8 ± 0.006	5/32 ± 0.008	3/16 ± 0.009

[1] All rivets to have fillet F under the head, max. radius 0.01 in. Heads of all rivets to have raised pimple for identification.

XXII, and XXIII give the data required to select the proper size of rivet for an assumed plate thickness.

277. Types and Dimensions of Rivets.—Figure 183 shows various types of aluminum and aluminum-alloy rivets in general use. The use and sometimes the appearance will determine the type. For example, the mushroom-head rivet would be desirable where the head is exposed to the air stream and where the sheet is very thin. The button-head would be more applicable where



Kind	Diam. A	Width B	Head Depth C	Head Radius R	Edge Radius E	Ctsk. Depth D	Tube Depth F
1. Button head	A	1.75A	0.75A	0.885A	..	..	..
2. Round head	A	2 A	0.75A	1.042A	..	..	..
3. Mushroom head	A	2 A	0.625A	1.634A	0.54A	..	..
4. Brazier head	A	2.50A	0.50A	1.8125A	..	..	..
5. Flat head	A	2 A	0.4A	..	..	..	..
6. Ctsk. head	A	1.81A	0.5A	..	..	..	..
7. Ctsk. oval head	A	2.50A	0.5A	1.8125A	..	0.75	..
8. Tubular shank	0.140"	0.375		..	..		0.25
	0.187"	0.4375					

FIG. 183.—Common types of rivets that may be made from aluminum and aluminum alloys. (*Courtesy of Aluminum Company of America; from The Riveting of Aluminum.*)

the rivet is subjected to a tensile load. The countersunk flat head would be applicable for thick plates over which other plates must fit snugly.

Tables XXIV and XXV give the S. A. E. standard for aluminum-alloy rivets.

278. Types of Riveting.—The types of riveting may be classified as follows:

1. Hand peening.
2. Automatic hammer riveting.
   a. Electric hammer.
   b. Air hammer.

c. Mechanical hammer.

3. Squeezing rivets in place.
   a. Hydraulic power.
   b. Air power.
   c. Mechanical power.

TABLE XXV.—S. A. E. STANDARD OF ALUMINUM-ALLOY RIVET LENGTHS[1]

Body diameter, in.	Length, in. ±0.010 in.																			
	3/16	1/4	5/16	3/8	7/16	1/2	5/8	3/4	7/8	1	1 1/8	1 1/4	1 3/8	1 1/2	1 3/4	2	2 1/2	3	3 1/2	4
1/16	X	X	X	X	X	X	X	X	X	X	X	X								
3/32	X	X	X	X	X	X	X	X	X	X	X	X	X							
1/8	X	X	X	X	X	X	X	X	X	X	X	X	X	X						
5/32	..	X	X	X	X	X	X	X	X	X	X	X	X	X	X					
3/16	..	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		
1/4	..	..	..	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	
5/16	..	..	..	..	..	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
3/8	..	..	..	..	..	..	X	X	X	X	X	X	X	X	X	X	X	X	X	X

[1] Report of the Aircraft Division adopted by the Society, January, 1931. Revised, June 1931.

Very little hand peening is done in aircraft work. There are various sizes of automatic hammers to fit the various sizes of rivets. The hammer is usually applied to the head of the rivet with the rivet in position through a block, stationary with respect to the rivet but attached to the hammer, while the dolly bar forms the cap on the shank end of the rivet. This practice makes riveting possible in places which would otherwise be inaccessible.

The squeeze-type riveter is quite generally used in connection with very accessible work, such as in riveting small parts which may be readily moved. This type of riveting is quite rapid and generally satisfactory.

279. Design for Accessibility in Riveting.—In designing a riveted structure, especially of the stressed-skin type, care must be used that the resulting design is sufficiently simple for efficient fabrication. Riveted joints should be made as simple and accessible as possible, consistent with reasonable efficiency in strength characteristics.

In the construction of closed sections, access must be provided to the inside for bucking the rivets which close the last opening.

280. Riveting to Small-diameter Tube.—The strength efficiency of a tube in thin sheet-metal construction, especially where

torsion is involved, outweighs the inefficiency of fabrication. Figure 184 shows the most common method of bucking rivets inside a tube. The steel blocks A and B slide on each other at the inclined surfaces MN when actuated by the rods C and D extending out the end of the tube. This sliding action causes the shank h to decrease as the cap is formed on the rivet.

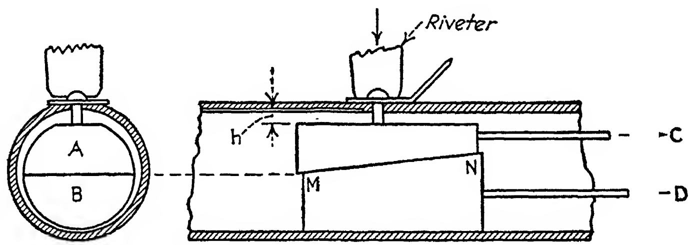


FIG. 184.—Riveting to the walls of a tube.

While there are many other devices for this purpose, it is sufficient for the designer to realize the possibilities so that he may not be handicapped in his design.

281. Allowance for Forming Rivet Heads.—The length of shank required for forming a rivet head depends upon the type of head and size of the rivet hole. The average is specified at about twice the diameter of the shank of the rivet.

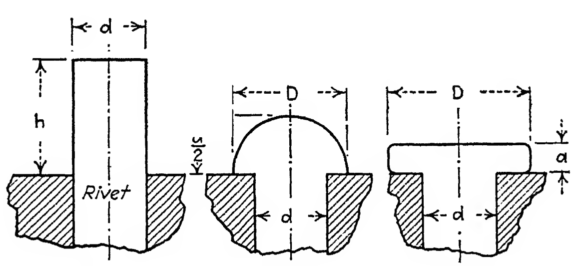


FIG. 185.—Shank required to form rivet head.

Upon the assumption of the size of the rivet hole and the size and type of rivet head to be made, the length of shank required may be calculated by the application of solid geometry. For example, in Fig. 185, the volume of the cylinder (shank) of diameter d and height h must equal the volume of the semi-sphere of diameter D. Thus

$$\frac{1}{12}\pi D^3 = \frac{\pi d^2}{4}h \tag{596}$$

If we require D to be $2d$, then, solving for h, we have

$$h = 2.6d \tag{597}$$

On the other hand, if the head is to be flat as in c, we have

$$\frac{\pi D^2}{4}a = \frac{\pi d^2}{4}h \tag{598}$$

If we require D to be $2d$, and a to be $\frac{1}{2}d$, we have

$$h = 2d.$$

We have assumed that the rivet exactly fits the rivet hole, and that the shank does not swell in driving except at the head.

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CHAPTER XVII

WELDING IN AIRCRAFT CONSTRUCTION

By N. F. Ward

282. Methods of Welding.—The principal methods of welding in aircraft construction are:

1. Fusion welding.
2. Resistance welding.

In the fusion-welding process metals of similar composition are joined by chemical union at their fusion heats without the application of external pressure. The sources of heat for fusion welding of airplane structures or fittings are the oxyhydrogen flame, the oxyacetylene flame, and the electric arc. This fusion process requires the addition of molten filler rod to supply the proper metal between adjoining interfaces at the junctures or seams. For thin-gage metals, the adjacent plates to be fused can readily be bent at right angles to the plate by an amount which is sufficient to fuse under the flame or arc heat and produce a bead of seam metal without the use of additional filler-rod metal.

The resistance-welding process differs from the fusion-welding method in that an external pressure is localized at a limited area of fusion. The process is often called *spot* or *shot* welding. A large magnitude of current, at low voltage, is conducted through adjoining metals which are under a compressive pressure; sufficient heat is developed with short-power impulses to fuse a limited area at the interfaces of the metals in contact. This type of welding is adaptable to thin-gage metals which are of different kind and composition. Only the welding characteristics of the adjacent metals must be known (see page 311).

283. Application of Fusion Welding.—The practice of fusion welding tubular fuselage and fittings has endured for many years. This type of welding still exists because it produces a reliable structural unit with favorable physical properties, a minimum of excess weight, and the advantage of ease of repair and maintenance.

With the advent of the modern all-metal fabrication, other methods, such as riveting and resistance welding, have displaced flame and arc welding. The lessened emphasis on fusion welding may be attributed principally to the vitiating effects of welding heats upon the alloys in the thin sections which now prevail, and the further difficulties encountered where alloys of low melting temperatures are required.

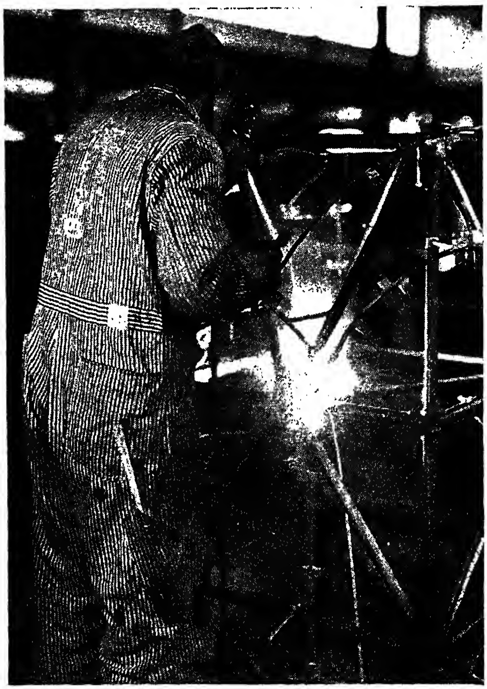

FIG. 186.- -Arc-welded nacelle for modern all-metal transport plane. (*Courtesy Boeing Airplane Company.*)

From the structural and economical standpoint, fusion welding is utilized effectively in the fabrication of such structural units as engine mounts, nacelles (Fig. 186), landing gears, and fittings. Exhaust stacks and fuel tanks are commonly welded by the fusion process. Since the gas and oil tanks are of aluminum composition with low melting temperatures, the oxyhydrogen flame is used because of its reduced temperature, although the oxyacteylene flame and electric arc are satisfactory when regulated to their lowest temperatures.

284. Characteristics of a Fusion-welded Joint.—A study of the failure of welded joints under static loading, such as is encountered in standard hardness and tension tests, although not simulated in actual service, serves, with the aid of microscopic examination, to detect inherent characteristics of welded joints. When these qualities in the weld and its immediate vicinity are recognized, suitable application of this type of joint is possible.

From specific data that are available[1] for tubes of chrome-molybdenum steel (S. A. E. 4130) which were welded by experienced welders with the oxyacetylene flame and low-carbon-steel welding rod, the welded tubes developed a tensile strength of 60 per cent ±5 per cent of the original tube strength. All failures occurred outside of the weld, except where misalignment during tests or imperfect welds caused failure in the weld (see Fig. 187). Appropriate heat treatment subsequent to welding, consisting of normalizing at 1700°F., hardening in water from 1650°F., and tempering at 900°F., developed a welded-tube strength which was 90 per cent of the original non-welded-tube strength.

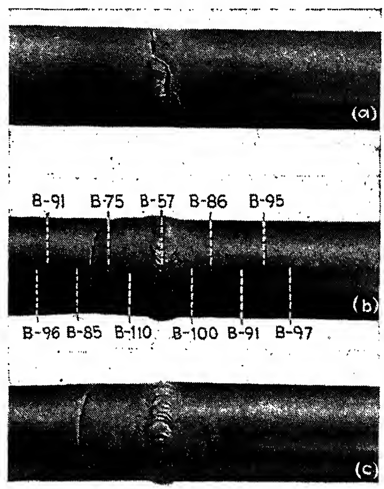

Fig. 187.—Performance of oxyacetylene welded tubes (S. A. E. 4130) under tensile loading. (*a*) Failure due to misalignment; (*b*) and (*c*) Rockwell hardness variation in oxyacetylene welded tubes and location of failure.

While the quality of the welds is dependent upon the training and experience of the welders executing them, the results are in close agreement, varying by ±5 per cent. In design this variation is adequately provided for by the usual *factor of safety*.

The ductility of these welds ranged from 60 to 71 per cent of the original because of the air-hardening properties of chrome-molybdenum steel. Only by the use of welded mild-steel tubing

[1] Johnson, J. B.: Development in Oxyacetylene Welding in Aircraft Industry, *Western Machinery World*, October, 1931.

Ward, N. F.: Reliability in Aircraft Welds, preprint, *A. S. M. E. Aeronautics Meeting*, June, 1934.

is the ductility satisfactory. However, the strength proves to be only half that of the chrome-molybdenum welded tubing.

The endurance limit of welded chrome-molybdenum tubing may be taken as one-fourth of the static tensile strength. After appropriate heat treatment, the endurance limit may be taken as one-third the static tensile strength of unwelded tubes.

Failure in welded chrome-molybdenum steel tubing which has been subjected to fatigue tests producing axial stress usually occurs at the juncture of the weld bead and the tube wall (Fig. 188). Heat treatment of welded tubes adjusts the microstructure so that the fracture with fatigue loading occurs most frequently at a distance from the weld equal to the width of the weld bead.

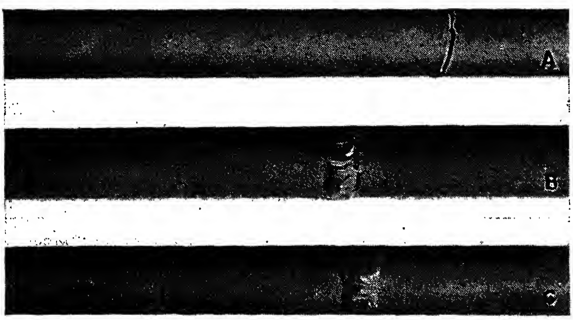

FIG. 188.—Fatigue tests of ½ inch chrome-molybdenum tubing. (*A*) Original tube failure; (*B*) and (*C*) oxyacetylene weld failure adjacent to bead.

Obviously, the desire in service loading is to preserve these critical sections against concentration of stresses; this may be accomplished by any one or more of several types of construction, such as *full-gusseting*, or *reinforcement with strap gussets*, or *telescoping tubes*, or use of a *fish-mouth joint* at the juncture of tubes.

285. Characteristics of an Arc-welded Joint.—Metallic arc welding with direct current and low-carbon-steel electrode has had limited application in airplane structures. For adequate fusion of the tube or plate the bead deposit is greater than for oxyacetylene welding. The concentrated heat from the arc localizes the critical zone in close proximity to the weld beads. The annealed section, which is less extensive than that for the oxyacetylene welds, develops within ⅜ in. of the welds. The

variation in hardness appears to be more marked than for the oxyacetylene tube.

286. Welding Stainless Steel.—This metal is utilized most economically in thin gages. The success of fusion welding either with the oxyacetylene flame or with the electric arc is largely dependent upon the design and the special skill of the welder. When the thickness of the sheet to be welded is small, the sheet cools rapidly. In consequence, the shrinkage stresses during cooling of the weld develop cracks during the time of welding or after the weld is completed, unless the design of the seam at the joint has the following provisions: Primarily, the joint must be accessible to permit rapid welding with the least possibility of overheating, since overheating embrittles the steel. Weld metal must be deposited evenly and the welds spaced as far apart as feasible to prevent warpage. Difficulty with cracking accompanies welding on both sides, because the cooler bead is stronger and concentrates the shrinkage stresses in the hot bead which is in the weaker condition. If the design requires welding on both sides, the weld should be staggered or alternated, which entails too much expense.

A greater separation of adjacent edges to be welded must be made than for mild steel, the amount of increase being 50 per cent, which corresponds to greater thermal contraction of stainless steel.

Flux is a necessary complement to the fusion welding of stainless steel and is most conveniently handled with the electric arc having a reversed polarity.

With either the gas-flame or the electric-arc method, the stainless steel suffers a loss of its strength and stainless property. The additional expense of fusion welding stainless steel in airplanes, as compared with the lesser cost of spot welding, is considered to be a deterrent in extending fusion welding to thin-gage stainless steel.

287. Welding Duralumin.—Duralumin is widely used in modern all-metal airplanes in thin gages. Joining of aluminum alloys by fusion welding is satisfactory with the oxyacetylene, oxyhydrogen, electric-arc, or resistance-welding equipment. Flux must be used in welding aluminum and magnesium alloys with the flame or arc to prevent oxidation and resultant lowering in strength and ductility. Removal of the flux is imperative to prevent its corrosive action on weld metal. Ductile welds of

highest strength require the heat treatment accorded similar alloys when not welded. Strength of welds and adjustment of temperature stresses, which are more severe than in steel because the contraction of aluminum alloys upon cooling is nearly double that of steel, require use of (1) a special 5 per cent silicon rod to prevent chilling and rapid shrinkage; (2) peening or working in the hot state at the completion of the weld; (3) heat treatment of welded sections to refine the grain size to that of parent metals and to adjust residual stresses in the weld and adjacent metal.

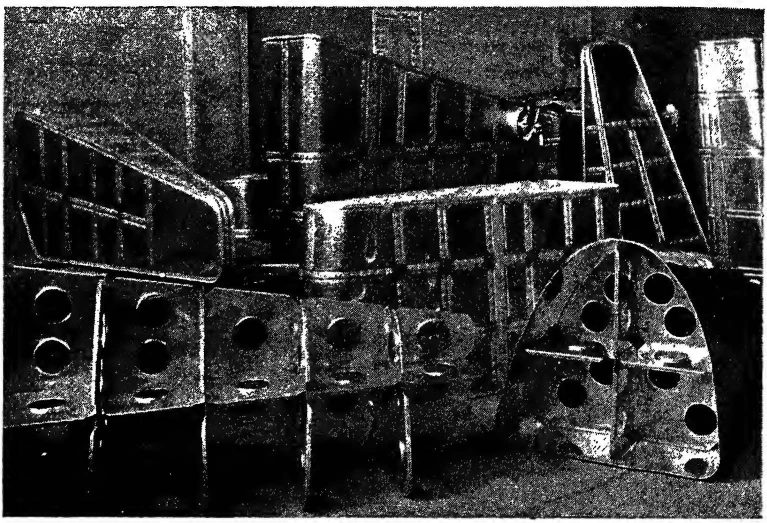

FIG. 189.—Aluminum fuel tanks. Welded edges, flat seams and rivet heads. (*Courtesy of W. H. Beall, Boeing Airplane Company.*)

It is inadvisable, considering the present progress of fusion welding, to weld thin sheets of aluminum alloy, since they are of low melting temperature and have a tendency to oxidize and burn at welding heats, thus producing poor homogeneity in the welded structural sections. *Riveting produces more reliable joints in this thin-sheet construction.*

Tanks and fittings of duralumin are most economically fabricated by welding. But to develop maximum properties of strength and corrosion resistance, heat treatment must follow in accordance with the specification of the alloy. Figure 189 shows welded tanks in different stages of completion. Rivets in these tanks are sealed by welding. Tanks which are fabricated by welding have capacities of 55 to 250 gallons.

288. Welding Strong Magnesium Alloys.—Sheet and extruded shapes are weldable with the oxyacetylene or oxyhydrogen flame. For heavy sections, the hotter flame is advisable because of the high thermal diffusivity in these sections. Light sections respond to the oxyhydrogen flame more advantageously. Welded sections must be cleaned to remove the flux because of its tendency to corrode welded parts.

289. Types of Magnesium-alloy Welds.—Butt seams are advisable for satisfactory fusion welds. Sketches of such welds

FIG. 190.—Recommended welds for magnesium alloy.

appear in Fig. 190. Lap or angle welds as shown in Fig. 191 are not recommended because the high concentration of temperature stress embrittles the casting as do also the least traces of salt flux and its corroding effects, which are difficult to remove in joints of this design.

Castings can be welded to sheet if sections of castings are reduced to the approximate sheet thickness by filing with a coarse file such as a vixen file. This is valuable where tank fittings are welded to sheet tanks. The welds as a general rule are of low ductility; where vibration is severe, riveting is advisable.

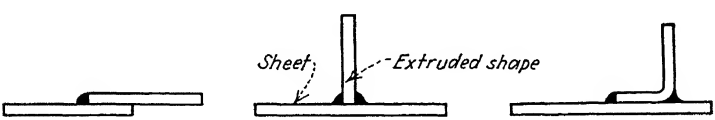

FIG. 191.—Improper welds for magnesium alloy.

290. Resistance Welding of Thin-gage Metals.—Resistance welding as a fabrication tool for thin-gage metal structures has been developed for a wide range of metal combinations (see Fig. 200). Recognition has been made of the possibilities for effecting economies in fabrication and speed of assembly which are inherent in the simplicity of design required for the resistance-welding process. The major difficulty encountered depends upon the electromechanical details of the process.

When high quality of weld metal is necessary, *accuracy of welding time, electrode pressure,* and *power impulse* are essential; uniformity of quality in a joint is more closely controlled by the resistance method than by fusion welding. The loss of strength for strong aluminum and stainless-steel alloys (which are the metals under discussion) is conditioned by these factors. In any event accurate evaluation is possible, whereas for the riveted joint many details are doubtful, especially the rivet hole and rivet shank relationship, except when determined by X-ray. In other words, the resistance-welded joint in its inherent refinement can be designed for accurate limits of loading when scientific control is maintained.

291. Factors Affecting Quality of Spot Welds.—A brief analysis shows that many variables of the process are involved which affect the physical and metallurgical properties of the spot welds. The accuracy with which these factors are controlled determines, in a large degree, the success or failure at the spot weld. The important considerations are outlined here:

1. *Heat generated for given current and resistance of specimen metal.*—The surfaces of the specimen must be clean to establish electrical contact. The capacity of the welder to produce large currents at low impressed voltage is inherent in the design of the welding unit available. Only by manipulation of time, *power impulse,* and *current taps* can optimum results be obtained.

2. *Pressure at electrodes.*—Aluminum alloys have little or no compressive strength at fusion temperatures. Electrode contact with the surfaces must be firmly established and kept cool at the electrode tip in order to localize fusion between interfaces of the metal and prevent serious undercutting of the areas in contact with the electrodes. When the pressure is insufficient to maintain electrical contact, arcing may occur which either fuses the sheet to the electrodes or leaves a porous weld.

3. *Electrode shape.*—Relatively small electrodes insure the required high-current density with the least disturbance of the magnetic field. A cylindrically shaped lower electrode with flat top and an upper electrode tip which is slightly spherical produce welds without undercutting when the electrode pressure is correctly adjusted. Tips which are refinished with a file leave their rough imprint on the specimen, and increase metal pickup by the electrode. Re-dressing with fine sandpaper produces the best finish, unless there is serious deformation.

4. *Time duration of power impulse.*—A prolonged power cycle produces a rapid deterioration in strength. A power impulse which is too short in duration, unless it is of high amperage, results in an imperfect bond between metals at the electrode tips. Accurate timing is essential for duplication of results.

5. *Condition of metal to be welded as affected by temperature.*—Duralumin maintains a surface without undercutting when current density is high,

electrode contact is cool, pressure is moderate, and time impulse is a minimum. Duralumin does not appear to age naturally after welding or to regain the strength at the weld. Subsequent heat treatment for such a small area as a spot weld involves additional handling and cost.

6. *Thickness of metal.*—Very thin gages cool quickly and develop higher unit strengths for the same type of joint (lap, butt with single-cover or double-cover plates). Special jigs are an essential part of equipment to insure alignment of adjacent parts. Elastic failure in misaligned parts intensifies the unit stress in the weld, which can least afford to be jeopardized. Heavy-gage metal localizes loads in the softer welds if allowed to weave in the structure. Double or triple rows of alternately spaced welds furnish necessary stiffness at the juncture of plates to intensify stress in multiple-spot welds. Tests show that the strength does not increase directly with the increase in number of spot welds.

7. *Length of secondary leads.*—Use of short leads reduces voltage drop to a minimum value for maximum current density at weld.

8. *Spacing of welds.*—Owing to the presence of adjoining spot welds, an electrical shunt exists which diverts a portion of the current from the weld being made. As much as 10 per cent additional current is required to produce an equivalent strength in welds. According to available research ½-in. spacing develops the best results in thin-gage welds.

TABLE XXVI.—LAP WELDS OF THIN-GAGE DURALUMIN (17ST)

Thickness of sheet for ⅝-in. width,	Electrode pressure		Welding time, sec.	Welding secondary current, amperes	Optimum Strength per Weld in Tension, Lb./Sq. In.	Efficiency of Strength of Metal Strip ⅝ In. Wide with One Weld, Per Cent
	Lb.	Lb./sq. in. of ⅛-in. electrode				
0.018 or No. 25 B. and S. gage..	66.5	5,400	0.045	11,200	15,450	31.5
0.020 or No. 24 B. and S. gage..	94.5	7,660	0.032	16,400	14,600	29
0.032 or No. 20 B. and S. gage..	135	11,200	0.029	20,000	11,200	23

Tables XXVI and XXVII for spot-welded joints serve as a guide in estimating requirements of this type of construction. These welds were made on automatic equipment under accurate control and were not heat treated after the welds had been completed.

From Table XXVII, the following conclusions are drawn: In the butt weld with double-cover plate, the two-spot fusion produces higher joint efficiency than three-spot fusion. Possibly the difference is due to the annealing effect of the additional heat in making the three spots or a greater number of spots per linear inch and the loss in available current in the shunt effect of adjacent welds.

TABLE XXVII.—STRENGTH OF SPOT WELDS IN ALCLAD 17ST SHEET[1]
(Specimens: 1 in. wide, 0.0385 in. thick)

Style of joint	Number of spots	Range in strength, lb.	Average strength, lb.	Efficiency of strength of metal strip 1 in. wide, per cent
Butt—double-cover plate		520–770	640	30
Butt—double-cover plate		,170–1,280	1,230	57
Butt—double-cover plate		,000–1,250	1,112	52
Lap joint...............		369–410	391	18
Lap joint...............		611–791	690	32
Lap joint...............		930–997	973	45

[1] From "Aluminum in Aircraft," Aluminum Company of America.

Removal of heat by wet asbestos or cold-air draft as the welds are completed tends to reduce the discrepancy in multiple-spot welding.

As yet structural spot welding for fuselage members has not been acceptable for airplane construction by the Aeronautics Branch of the United States Department of Commerce. Riveted joints remain superior in reliability and durability. Such units as ventilators, minor fittings, and instrument housings are commonly spot welded.

292. Shot Welding.—This is the name of a spot-welding process developed by the Edward G. Budd Manufacturing Company of Philadelphia and is applied most effectively to stainless steel. When resistance welding was applied to thin sections of stainless steel, it was found to cause segregation of the chromium at the welded joints, poor immunity to corrosion, and reduction of strength and fatigue resistance. The "shot" welding process develops very high temperatures in less time than for usual spot welding so that there was no chromium separation. The time during which the current is applied varies from 0.010 to 0.03 sec. with amperage as high as 12,000. The

rapidity with which the heat is applied to lap and seam joints precludes the occurrence of detrimental metallurgical changes.

293. Other Factors Affecting Reliability of Welded Joints.—A failure due to excessive reinforcement of the bead sometimes results. The cause of failure may be attributed to (1) abrupt change in section, and (2) concentration of temperature stress. Contraction or shrinkage, when cooling at the weld takes place, localizes the stresses at the fused junctures; this must be adjusted in the latter areas first since these are the more ductile. The acceptable proportions for satisfactory welds are given in Fig. 192.

For instance, if a bar of metal is held rigidly at its ends (this condition obtains for certain aircraft members welded or heated in jigs) so as to prevent expansion or contraction, stresses are

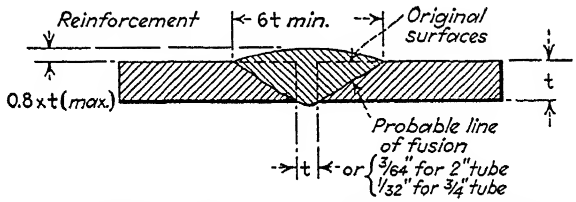

FIG. 192.—Typical airplane weld in cross section with dimensions for guiding designer.

produced which are known as *temperature stresses.* Existing of temperature stresses depends solely upon expansion and contraction of parts when heated. The effect is the same as though the bar had been compressed an amount equivalent to the expansion under welding heat, or had been elongated an amount equal to the shrinkage of the heated bar after cooling. The actual movement of the weld metal is proportional to the coefficient of linear expansion.

It is evident that the temperature stresses due to elongation or shrinkage of metals being welded vary directly with temperature. Hence if the weld is placed at 3100°F. (a fair estimate of usual bead temperature of the arc weld), and the metal in the original plate is 1000°F., the weld and its adjacent volumes must shrink about three times that of the original plate. The metal which is hottest has the least strength and must absorb the contraction upon cooling. For the gas weld, the overheated section is greater than that for the arc weld, hence the gas weld encounters

less stress intensity than the arc weld since the annealed section of the arc weld is closer to the weld and therefore yields first.

In many cases where no elaborate methods are taken to adjust or distribute temperature stresses, failure is prevented by superior strength and ductility of the steels being welded.

In the welding of castings, which are not so ductile as rolled steel, gradual cooling of the welded portion is essential to the success of the weld. The slightest amount of localized temperature stress will leave incipient cracks and hasten failure.

Other defects in welds must be guarded against to insure reliable joints. These are (1) presence of oxygen, and (2) presence of excess carbon, which produces hard welds of low impact resistance. Since both are results of welding technique, welders who are adequately trained should prepare the welds.

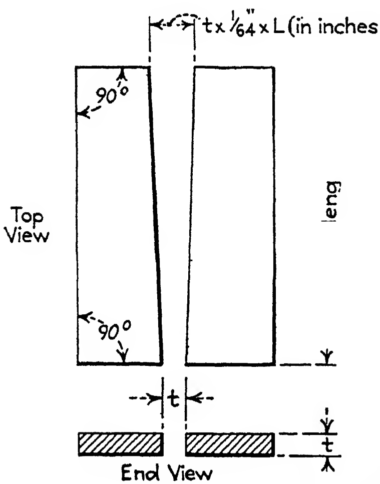

FIG. 193.—Allowance for shrinkage in steel plate welding.

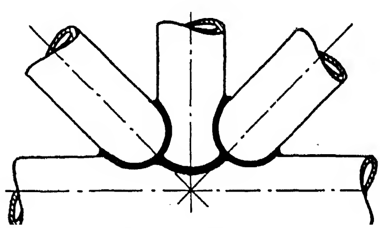

FIG. 194.—Cluster weld.

Oftentimes the steel may have defects, and, no matter how excellently trained the welder is, the weld will suffer.

294. Design of Welded Joints.—As for the designer's knowledge of these internal adjustments, his design must provide for proper spacing in which to weld without producing excessive thermal stresses. Plates are usually spaced as shown in Fig. 193. The constant $\frac{1}{64} \times L$ is based upon and applies to steel and an average weld temperature of 3000°F. L is expressed in inches. Tubes are mounted with their axes in alignment (Figs. 194, 195, and 196) or in a cluster, with their axes intersecting at the center line of the longeron and the surfaces at the juncture parallel and $\frac{3}{64}$ to $\frac{1}{32}$ in. apart. This applies also to tubes welded to plates.

Seams in aluminum sheets, as, for example, in tanks, must provide for a contraction twice that of steel. The usual design for welds in aluminum requires methods in which flexibility on either side of the seam is inherent in the construction. For example, butt welds are joined on a flexible seam which is shown in Fig. 197*a*. Similar construction is used for a corner weld as in

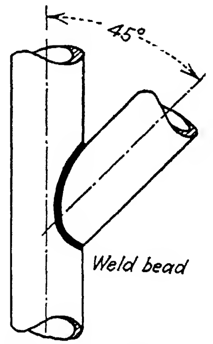

FIG. 195.—Angle tube weld.

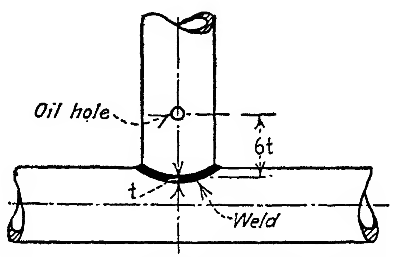

FIG. 196.—Location of oil hole in welded tubes.

Fig. 197*b*. Often notched plates are made use of as illustrated in Fig. 198*a*. The notches serve as repositories for the flux required to prevent oxidation of the aluminum during welding, and permit contraction to be absorbed, thus relieving localized distortion on the aluminum, which is very brittle at welding heats. For aluminum sheet of extremely small-gage thickness, the edges

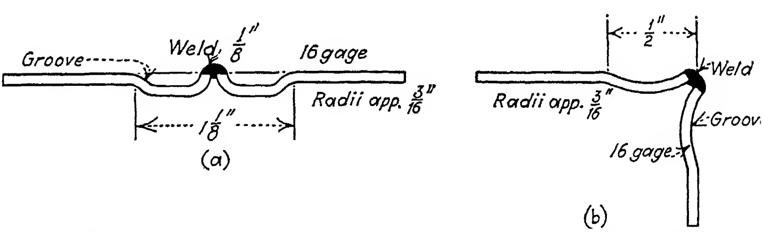

FIG. 197.—(*a*) Butt joint in aluminum sheet; (*b*) corner weld in aluminum.

are flanged and welded with flux coated on the seam, allowing the flange to fuse into the seam. Proportions of the joint are listed in Fig. 198*b*.

295. Precautions in Welded-joint Design.—The practice of placing solid gussets between tubes which intersect at an angle of less than 30 degrees concentrates large crushing tensile and shear

stresses at the welded joints, and produces buckling tendencies in the softened area of the tube or plate. Often no gussets are used. If gussets are used in these locations, strap-gussets as

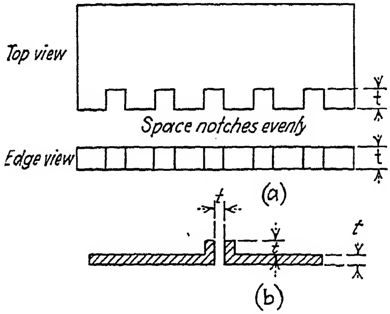

FIG. 198.—(a) Notched plate for aluminum over 1/4 in. thick; (b) flanged butt joint for aluminum plate of sheet under 1/8 in. in thickness. No filler rod required.

shown in Fig. 199 are valuable when the tubes are subjected to impact loading. Gussets that are extended to the juncture of tubes are of doubtful value where welding is used on tubes intersecting at less than 45 degrees.

FIG. 199.—Showing effectiveness of strap-gusset plates. Tubes bent but welds held. (*Courtesy of Linde Air Products Company.*)

Chrome-molybdenum steel is of less advantage than mild steel, because of its low ductility, where vibration is a consideration, as, for example, in engine mounts. For similar reasons alloy-steel wire seems unnecessary for chrome-molybdenum

steel welding because of its tendency to produce welds of greater hardness and strength than the tube, with further concentration of stress at the juncture of weld and tube or plate. The best method for producing a uniform distribution of weld and tube strength lies in adequate heat treatment of the welded structure.

296. Welding Jigs.—The duplication of welded structures in airplane assemblies is facilitated by the use of jigs. The structure of a jig conforms to the shape of the parts to be held rigidly while being welded. The requirements of welding jigs are:

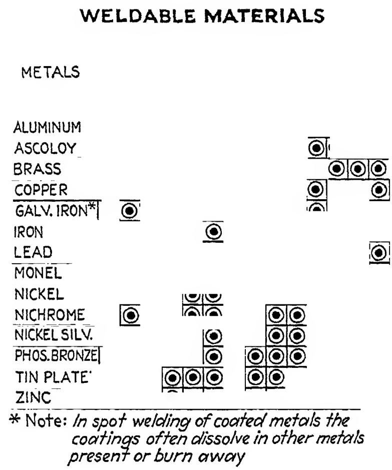

Fig. 200.—A few of the many similar and dissimilar metals that will spot weld. (*Courtesy of P. F. Mallory Company and R. T. Gillette.*)

1. Hold parts in alignment during the welding process.
2. Permit expansion and contraction of welded parts within the allowable tolerances.
3. Retain shape during welding; that is, have rigidity and strength so that warping does not occur in the jigs themselves.
4. Insure accuracy in welded assemblies.
5. Produce duplicate welded assemblies which make possible interchangeableness of original and repair parts.

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CHAPTER XVIII

LANDING-GEAR STRESS ANALYSIS

By R. H. Rice

297. The Design Problem.—The stress analysis of landing gears has always been a rather complex problem although the principles involved are fairly simple. The object of this chapter is to present a method for determining loads in the members, which is accurate yet involves several short cuts that reduce the amount of labor to a minimum.

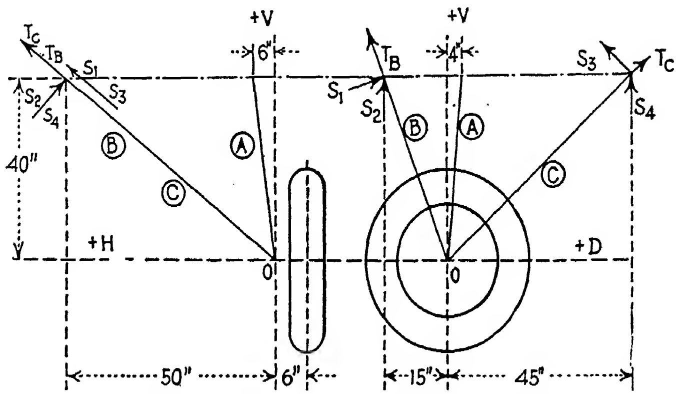

FIG. 201.—Free-body diagram of landing gear.

Most landing gears are similar in appearance in that they are generally composed of two separate tripods, but each offers its own particular problem owing to the different manner in which the members may be fastened together at the apex and the nature of the fittings at the fuselage. Figure 201 shows a front and side elevation of a typical chassis. Member *A* contains the shock-absorbing mechanism and may be compressed or extended. The fitting at the apex is such that it cannot resist the applied loads by bending but only by an axial load. Members *B* and *C* are hinged about a line connecting their upper ends.

298. Cases to be Considered.—Three typical cases will be considered when the fixity of members *B* and *C* differs.

Case I.—Members C and A are connected to the apex by a fitting that cannot transmit torsion or bending. This means that member B must resist all applied couples by torsion and bending.

Case II.—Members B and C are fastened solidly together at the apex either by welding or by bolting to a common fitting. The fittings at the fuselage are such that they cannot transmit bending or torsion. This means that members B and C will resist all applied couples by bending only.

Case III.—Members B and C are fastened solidly together at the apex, as in Case II, but the fittings at the fuselage are such that they can resist torsion. Applied couples will, therefore, be resisted by both torsion and bending in each member.

299. Free-body Diagrams.—For the first step in each case we must draw a free-body diagram of the complete structure; that is, if the body is in space and the loads applied, the torsional, shear, and direct reactions on the members must be such that the body will be in equilibrium. The *direct reactions* are, by definition, forces coincident with the axis of the members, while the *shear reactions* are forces perpendicular to the axis of the members. Since we do not know the direction of the resultant shear reaction on any one member, it will be convenient to assume the direction of two components of the shear reaction. For example, the two components of shear at the upper end of member B are S_1 and S_2. It is, of course, possible to assume any number of directions, but those shown in Fig. 201 have been found most convenient to use. Since a couple may be represented by a vector perpendicular to the plane in which the couple acts, the torsional reactions at the ends of members B and C are represented by a vector coincident with the axis of the members. The direction of the vectors is determined by the right-hand rule which states that if the fingers of the right hand point in the direction of the couple, the thumb will indicate the direction of the vector.

300. The Coordinate Axes.—The origin of the three reference axes is at the apex of the tripod. The D-axis is parallel to a line joining the upper ends of members B and C, which in turn happens to be parallel to the propeller axis. The V-axis is perpendicular to the propeller axis and parallel to the plane of symmetry. The H-axis is perpendicular to the V- and D-axes. In cases where the hinge line and the propeller axis are not parallel, it will

usually be found more convenient to make the D-axis parallel to the hinge line.

301. Designation of Reactions.—The reactions are:

S_1 = a shear reaction lying in the plane of members B and C, perpendicular to member B and applied at its upper end.

S_2 = a shear reaction perpendicular to the plane of members B and C and applied at the upper end of member B.

S_3 = a shear reaction lying in the plane of members B and C, perpendicular to member C and applied at its upper end.

S_4 = a shear reaction perpendicular to the plane of members B and C and applied at the upper end of member C.

T_B = a torsional reaction at the end of member B.

T_C = a torsional reaction at the end of member C.

P_A, P_B, and P_C will represent the direct reactions on members A, B, and C respectively. All reactions that slope upward are taken as being positive in sign. Thus a positive direct reaction will mean tension, and a negative one compression, in the member.

We assume in all cases that the fittings at the fuselage cannot transmit bending in any plane. This, of course, is not true where the fitting is not a universal joint, but the error thus incurred is negligible.

302. Direction Cosines of Members.—It will now be necessary to determine the direction cosines of the members and shear reactions with respect to the three reference axes. The direction cosine of a member is the projection of a unit length of a member on a given reference axis (see Table XXVIII).

TABLE XXVIII.—DIRECTION COSINES OF MEMBERS

Member	V, inches	D, inches	H, inches	L, inches	$\frac{V}{L}$	$\frac{D}{L}$	$\frac{H}{L}$
A	40	4	6	40.64	0.9843	0.0984	0.1476
B	40	15	50	65.76	0.6083	0.2281	0.7603
C	40	45	50	78.26	0.5111	0.5750	0.6389

$$L = \sqrt{V^2 + D^2 + H^2}$$

303. Direction Cosines of Shear Vectors.—Since we now know the direction cosines of the members and the direction of the shears with respect to the members, we can find the direction cosines of the shear reactions.

Projection of members B and C on the

$$V\text{-}H \text{ plane} = \sqrt{40^2 + 50^2} = 64.03 \text{ in.}$$

$$\frac{D}{S_2} = 0$$

$$\frac{V}{S_2} = \frac{50}{64.03} = 0.7809$$

$$\frac{H}{S_2} = \frac{40}{64.03} = 0.6247$$

Since S_4 is parallel to S_2, the direction cosines of these two shear reactions are the same.

$$\frac{D}{S_1} \quad \frac{64.03}{65.76} = 0.9737$$

Projection of unit S_1 on $V\text{-}H$ plane,

$$\frac{15}{65.76} = 0.2281.$$

$$\frac{V}{S_1} = 0.2281 \times \frac{40}{64.03} = 0.1425$$

$$\frac{H}{S_1} = 0.2281 \times \frac{50}{64.03} = 0.1781$$

In a similar manner,

$$\frac{D}{S_3} \quad \frac{64.03}{78.26} \quad 0.8182$$

Projection of unit S_3 on the $V\text{-}H$ plane,

$$\frac{45}{78.26} = 0.5750$$

$$\frac{V}{S_3} = 0.575 \times \frac{40}{64.03} = 0.3592$$

$$\frac{H}{S_3} = 0.575 \times \frac{50}{64.03} = 0.4490$$

These cosines are summarized in Table XXIX.

The sum of the squares of the direction cosines must equal 1. *It is advisable always to use this relation as a check.*

304. Equations of Equilibrium.—*Case I.*—We are now ready to solve for the reactions due to unit loads. To solve first for the reactions due to unit loads, and later to multiply these reactions by components of the actual applied loads saves much

repetition since the chassis must be analyzed for several loading conditions.

TABLE XXIX.—DIRECTION COSINES OF SHEARS

Shear	$\frac{V}{S}$	$\frac{D}{S}$	$\frac{H}{S}$
S_1	0.1425	0.9737	0.1781
S_2	0.7809	0.0000	0.6247
S_3	0.3592	0.8182	0.4490
S_4	0.7809	0.0000	0.6247

We assume that

$$S_3 = 0, \qquad S_4 = 0, \qquad T_C = 0$$

We now have six unknowns to solve for any system of applied loads, namely,

$$S_1, \quad S_2, \quad T_B, \quad P_A, \quad P_B \quad \text{and} \quad P_C$$

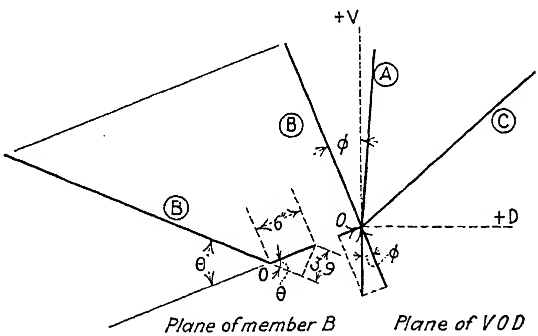

FIG. 202.

From mechanics we have the following six equations of static equilibrium; hence the structure in this case is statically determinate.

$$\Sigma V = 0, \qquad \Sigma D = 0, \qquad \Sigma H = 0$$

and $$\Sigma M_V = 0, \qquad \Sigma M_D = 0, \qquad \Sigma M_H = 0$$

The solution of six simultaneous equations would be quite tedious and is unnecessary if we do not confine ourselves to taking moments about the reference axes only. The moments

taken about any axis must be zero. If we make use of this fact, we can reduce the number of simultaneous equations to be solved to a minimum. In this case we will choose our axes in such a manner that there will be no simultaneous equations to solve, each equation giving one unknown directly. Some of the equations may appear rather long but are not difficult to set up if one has a clear mental picture of the free-body diagram.

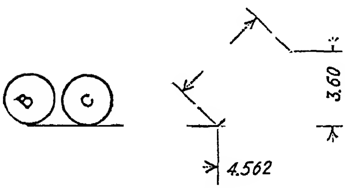

FIG. 203.

305. Case I, Solution for Unit Vertical Load.—We note in Figs. 201 and 202, that

$$\sin \phi = \frac{15}{\sqrt{40^2 + 15^2}} \quad \frac{15}{42.72} = 0.3511$$

$$\cos \phi = \frac{40}{42.72} = 0.9363$$

Taking moments about member B,

$$\Sigma M_B = 1 \times \sin \phi \times 3.9 + T_B = 0$$
$$1 \times 0.3511 \times 3.9 + T_B = 0$$
$$T_B = -1.369 \text{ in.-lb.}$$

Taking moments about the D-axis,

$$\sum M_D = -T_B \times \frac{D}{L_B} - 1 \times 6 + S_2 \times 64.03 = 0$$
$$+1.369 \times 0.2281 - 6 + 64.03\, S_2 = 0$$
$$S_2 = +\frac{5.688}{64.03} = +0.0888 \text{ lb.}$$

Taking moments about the H-axis,

$$\sum M_H = +S_2 \times \frac{V}{S_2} \times B_D + T_B \times \frac{H}{L_B} + S_1 \times \frac{D}{S_1} \times B_V + S_1 \times \frac{V}{S_1} \times B_D \quad 0$$

$$+0.0888 \times 0.7809 \times 15 - 1.369 \times 0.7603 + S_1 \times 0.9737 \times 40 + S_1 \times 0.1425 \times 15 = 0$$
$$+1.040 - 1.040 + 38.95 S_1 + 2.138 S_1 = 0$$
$$S_1 = 0$$

Taking moments about an axis through the upper ends of members B and C,

$$\sum M_{BC} = -(B_H + 6) \times 1 - T_B \times \frac{D}{L_B} - P_A + \frac{V}{L_A} \times B_H + P_A \times \frac{H}{L_A} \times B_V = 0$$

$$-(50 + 6) \times 1 + 1.369 \times 0.2281 - P_A \times 0.9843 \times 50 + P_A \times 0.1475 \times 40 = 0$$

$$-56 - 49.21\, P_A + 5.90\, P_A + 0.3123 = 0$$

$$P_A = -\frac{55.69}{43.31} = -1.286 \text{ lb.}$$

Taking moments about an axis through the upper end of member B and parallel to the H-axis,

$$\sum M_{BH} = +T_B \times \frac{H}{L_B} - P_A \times \frac{V}{L_A} \times B_D - P_A \times \frac{D}{L_A} \times B_V - 1 \times B_D - P_C \times \frac{V}{L_C} \times (B_D + C_D) = 0$$

$$-1.369 \times 0.7603 + 1.286 \times 0.9843 \times 15 + 1.286 \times 0.0984 \times 40 - 1 \times 15 - P_C \times 0.5111 \times (15 + 45) = 0$$

$$-1.040 + 18.99 + 5.06 - 15 - 30.67 \times P_C = 0$$

$$P_C = +\frac{8.01}{30.67} = +0.2612 \text{ lb.}$$

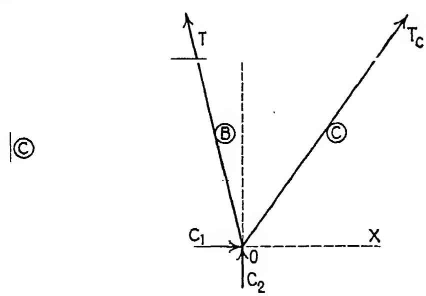

FIG. 204.

Taking moments about an axis through the upper end of member C and parallel to the H-axis,

$$\sum M_{CH} = +T_B \times \frac{H}{L_B} + P_A \times \frac{V}{L_A} \times C_D - P_A \times \frac{D}{L_A} \times C_V + S_2 \times \frac{V}{S_2} \times (B_D + C_D) + S_1 \times \frac{V}{S_1} + (B_D + C_D) + 1 \times C_D + P_B \times \frac{V}{L_B} \times (B_D + C_D) = 0$$

$$-1.369 \times 0.7603 - 1.286 \times 0.9843 \times 45 + 1.286 \times 0.0984 \times 40 + 0.0888 \times 0.7809 \times (15 + 45) + 0 + 1 \times 45 + P_B \times 0.6083 \times (15 + 45) = 0$$

$$-1.040 - 56.96 + 5.06 + 4.16 + 45 + 36.5 \times P_B = 0$$

$$P_B = +\frac{3.78}{36.5} = +0.1036 \text{ lb.}$$

306. Case I, Solution for Unit Drag Load.—Refer to Figs. 201 and 202.

Taking moments about member B,

$$\Sigma M_B = 1 \times \cos \phi \times 3.9 + T_B = 0$$

$$1 \times 0.9363 \times 3.9 + T_B = 0$$

$$T_B = -3.652 \text{ in.-lb.}$$

Taking moments about the H-axis,

$$\sum M_H = +S_2 \times \frac{V}{S_2} \times B_D + T_B \times \frac{H}{T_B} + S_1 \times \frac{D}{S_1} \times B_V + S_1 \times \frac{V}{S_1} \times B_D = 0$$

$$-0.013 \times 0.7809 \times 15 - 3.652 \times 0.7603 + S_1 \times 0.9737 \times 40 + S_1 \times 0.1425 \times 15 = 0$$

$$-0.1523 - 2.777 + 39.95\, S_1 + 2.137\, S_1 = 0$$

$$S_1 = +\frac{2.929}{42.09} \times +0.0696 \text{ lb.}$$

Taking moments about an axis through the upper ends of members B and C,

$$\sum M_{BC} = -T_B \times \frac{D}{L_B} - P_A \times \frac{V}{L_A} \times B_H + P_A \times \frac{H}{L_A} \times B_V = 0$$

$$+3.652 \times 0.2281 - P_A \times 0.9843 \times 50 + P_A \times 0.1476 \times 40 = 0$$

$$+ 0.8330 - 49.21\, P_A + 5.90\, P_A = 0$$

$$P_A = +\frac{0.8330}{43.31} \quad +0.0192 \text{ lb.}$$

Taking moments about an axis through the upper end of B and parallel to the H-axis,

$$\sum M_{BH} = +T_B \times \frac{H}{L_B} - P_A \times \frac{V}{L_A} \times B_D - P_A \times \frac{D}{L_A} \times B_V - 1 \times B_V - P_C \times \frac{V}{L_C} \times (B_D + C_D) = 0$$

$$-3.652 \times 0.7603 - 0.0192 \times 0.9843 \times 15 - 0.0192 \times 0.0984 \times 40 - 1 \times 40 - P_C \times 0.5111 \times (15 + 54) = 0$$

$$-2.777 - 0.284 - 0.076 - 40 - 30.67 \times P_C = 0$$

$$P_C = -\frac{43.136}{30.67} = -1.406 \text{ lb.}$$

Taking moments about an axis through the upper end of member C and parallel to the H-axis,

$$\sum M_{CH} = +T_B \times \frac{H}{L_B} + P_A \times \frac{V}{L_A} \times D_D - P_A \times \frac{D}{L_A} \times C_V + S_2 \times \frac{V}{S_2} \times (B_D + C_D) + S_1 \times \frac{V}{S_1} \times (B_D + C_D) - 1 \times B_V + P_B \times \frac{V}{L_B} \times (B_D + C_D) = 0$$

$$-3.652 \times 0.7603 - 0.0192 \times 0.9843 \times 45 - 0.0192 \times 0.0984 \times 40 - 0.013 \times 0.7809 \times (15 + 45) + 0.0696 \times 0.1425 \times (15 + 45) - 1 \times 40 + P_B \times 0.6083 \times (15 + 45) = 0$$

$$-2.777 + 0.850 - 0.076 - 0.6091 - 0.5951 - 40 + 36.5 \times P_B = 0$$

$$P_B = +\frac{42.016}{36.5} = +1.151 \text{ lb.}$$

For a complete analysis it would also be necessary to solve for the reactions due to unit side load acting at the ground and unit brake torque acting about the H-axis. These will not be carried out here as they are only repetitions of the above two examples.

307. Equations of Equilibrium.—Case II.—In this case

$$T_B = 0 \qquad \text{and} \qquad T_C = 0$$

We now have seven unknowns to solve for any system of applied loads, namely,

$$S_1, \quad S_2, \quad S_3, \quad S_4, \quad P_A, \quad P_B, \quad \text{and} \quad P_C$$

To solve for these unknowns we have only six equations of static equilibrium; hence the structure is statically indeterminate and we must have an additional equation of equilibrium which is based on the property of the members. However, if the applied couple is perpendicular to the plane of members

B and C, two of the unknowns are equal to zero and the structure is statically determinate. It is statically indeterminate if the applied couple is in the plane of members B and C for either member could resist the couple by bending without the aid of the other. Hence we may employ a very useful relation. The two members will resist the couple by a bending moment in each that is directly proportional to its moment of inertia and inversely proportional to its length

$$\frac{M'_B}{M'_C} = \frac{I_B}{I_C} \times \frac{L_C}{L_B} \tag{599}$$

Hence we have a seventh equation and the balancing reactions may be determined. In equation (599) M'_B and M'_C are bending moments in members B and C respectively at the apex and in the plane of the members.

308. Case II, Solution for Unit Vertical Load.—This is a case where the applied couple is perpendicular to the plane of members B and C,

$$S_1 = 0 \qquad \text{and} \qquad S_3 = 0$$

Taking moments about the D-axis,

$$\Sigma M_D = +64.03 S_2 + 64.03 S_4 - 1 \times 6 = 0 \tag{600}$$

Taking moments about the H-axis,

$$\sum M_H = +S_2 \times \frac{V}{S_2} \times B_D - S_4 \times \frac{V}{S_4} \times C_D = 0$$

$$S_2 \times 0.7809 \times 15 - S_4 \times 0.7809 \times 45 = 0$$

$$S_2 = 3S_4 \tag{601}$$

Solving (600) and (601),

$$S_2 = +0.0702 \text{ lb.} \tag{602}$$

$$S_4 = +0.0234 \text{ lb.} \tag{603}$$

Taking moments about an axis through the upper ends of members B and C,

$$\sum M_{BC} = -(B_H + 6) \times 1 - P_A \times \frac{V}{L_A} \times B_H + P_A \times \frac{H}{L_A} \times B_V = 0$$

$$-(50 + 6) - P_A \times 0.9843 \times 50 + P_A \times 0.1476 \times 40 = 0$$

$$-56 - 49.21\, P_A + 5.90\, P_A = 0$$

$$P_A = -\frac{56}{43.32} = -1.293 \text{ lb.} \tag{604}$$

Taking moments about an axis through the upper end of member B and parallel to the H-axis,

$$\sum M_{BH} = -B_D \times 1 - P_A \times \frac{V}{L_A} \times B_D - P_A \times \frac{D}{L_A} \times B_V - S_4 \times \frac{V}{S_4} \times (B_D + C_D) - P_C \times \frac{V}{L_C} \times (B_D + C_D) = 0$$

$$-15 \times 1 + 1.293 \times 0.9843 \times 15 + 1.293 \times 0.0984 \times 40 - 0.0234 \times 0.7809 \times (15 + 45) - P_C \times 0.5111 \times (15 + 45) = 0$$

$$-15 + 19.09 + 5.09 - 1.10 - 30.67 \times P_C = 0$$

$$P_C = +\frac{8.08}{30.67} = +0.2634 \text{ lb.} \tag{605}$$

Taking moments about an axis through the upper end of member C and parallel to H-axis,

$$\sum M_{CH} = +B_D \times 1 + P_A \times \frac{V}{L_A} \times C_D - P_A \times \frac{D}{L_A} \times C_V + S_2 \times \frac{V}{S_2} \times (B_D + C_D) + P_B \times \frac{V}{L_B} \times (B_D + C_D) = 0$$

$$-45 \times 1 - 1.293 \times 0.9843 \times 45 + 1.293 \times 0.0984 \times 40 + 0.0702 \times 0.7809 \times 60 + P_B \times 0.6083 \times (15 + 45) = 0$$

$$+45 - 57.27 + 5.09 + 3.29 + 36.5 \times P_B = 0$$

$$P_B = +\frac{3.89}{36.5} = +0.1066 \text{ lb.} \tag{606}$$

309. Case II, Solution for Unit Drag Load.—In this case $P_A = 0$.

For drag loads the structure is statically indeterminate and we must make use of equation (599). Member B, we will assume, is a round steel tube $2\frac{1}{4}$ in. in diameter with $\frac{3}{16}$-in. wall thickness, and member C is also a round tube of the same diameter but with a $\frac{5}{32}$-in. wall thickness. If we were to carry through a complete analysis and find our assumptions were not correct, it would be necessary to make new assumptions and go through the process again. (It is seldom necessary to repeat the process more than twice.) Per-

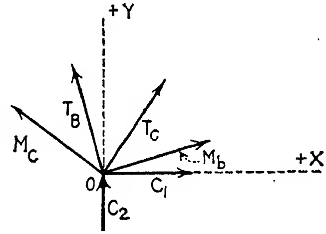

FIG. 205.

haps a better method would be to assume a ratio between I_B and I_C for the first approximation.

$$I_B = 0.6514 \text{ in.}^4 \qquad L_B = 65.76 \text{ in.}$$
$$I_C = 0.5663 \text{ in.}^4 \qquad L_C = 78.26 \text{ in.}$$
$$\frac{-M'_B}{M'_C} = \frac{S_1 \times L_B}{S_3 \times L_C} = \frac{I_B}{I_C} \times \frac{L_C}{L_B}$$
$$S_1 = \frac{-I_B}{I_C} \times \frac{L_C^2}{L_B^2} \times S_3$$
$$-\frac{0.6514}{0.5663} \times \frac{(78.26)^2}{(65.76)^2} \times S_3 = -1.629S_3 \qquad (607)$$

Taking moments about an axis through the origin and perpendicular to the plane of members B and C,

$$\Sigma M_O = +4.685 \times 1 - S_1 \times L_B + S_3 \times L_C = 0$$
$$+4.685 - S_1 \times 65.76 + S_3 \times 78.26 = 0 \qquad (608)$$

Solving (607) and (608),

$$S_1 = -0.0412 \text{ lb.} \qquad (609)$$
$$S_3 = -0.0253 \text{ lb.} \qquad (610)$$

Taking moments about the D-axis,

$$\Sigma M_D = 64.03S_2 + 64.03 \times S_4 = 0$$
$$S_2 = -S_4 \qquad (611)$$

Taking moments about the H-axis,

$$\sum M_H = +S_1 \times \frac{V}{S_1} \times B_D + S_1 \times \frac{D}{S_1} \times B_V - S_3 \times \frac{V}{S_3} \times C_D - S_3 \times \frac{D}{S_3} \times C_V - S_2 \times \frac{V}{S_2} \times B_D - S_4 \times \frac{V}{S_4} \times C_D = 0$$
$$+0.0412 \times 0.1425 \times 15 + 0.0412 \times 9737 \times 40 + 0.0253 \times 0.3592 \times 45 + 0.0253 \times 0.8182 \times 40 + S_2 \times 0.7809 \times 15 - S_4 \times 0.7809 \times 45 = 0$$
$$+.0881 + 1.6404 + 0.4089 + 0.8280 + 11.71S_2 - 35.14S_4 + 11.71\ S_2 - 35.14S_4 - 2.9296 = 0 \qquad (612)$$

Solving (611) and (612),

$$S_2 = -0.0625 \text{ lb.} \qquad (613)$$
$$S_4 = +0.0625 \text{ lb.} \qquad (614)$$

Taking moments about an axis through the upper end of member B and parallel to the H-axis,

$$\sum M_{BH} = -B_V \times 1 - S_4 \times \frac{V}{S_4} \times (B_D + C_D) - S_3 \times \frac{V}{S_3} \times (B_D + C_D) - P_C \times \frac{V}{L_C} \times (B_D + C_D) = 0$$

$$-1 \times 40 - 0.0625 \times 0.7809 \times (15 + 45) + 0.0253 \times 0.3592 \times (15 + 45) - P_C \times 0.5111 \times (15 + 45) = 0$$

$$-40 - 2.9284 + 0.5453 - 30.67 \times P_C = 0$$

$$P_C = -\frac{42.38}{30.67} = -1.3819 \text{ lb.}$$

Taking moments about an axis through the upper end of member C and parallel to the H-axis,

$$\sum M_{CH} = -B_V \times 1 + S_2 \times \frac{V}{S_2} - (B_D + C_D) + S_1 \times \frac{V}{S_1} \times (B_D + C_D) + P_B \times \frac{V}{L_B} \times (B_D + C_D) = 0$$

$$-1 \times 40 - 0.0625 \times 0.7809 \times (15 + 45) + 0.0412 \times 0.1425 \times (15 + 45) + P_B \times 0.6083 \times (15 + 45) = 0$$

$$-40 - 2.9284 + 0.3523 + 36.5 \times P_B = 0$$

$$P_B = +\frac{42.576}{36.5} = +1.1664 \text{ lb.}$$

310. Application of Cases I and II to Design Loads.—Assume the following data:

Gross weight of airplane = 7,500 lb.
Weight of landing gear = 375 lb.
Landing load factor = 5.35

With reference to Fig. 206,

$$\Sigma M_{R_1} = 32 \times 7{,}500 - 292 R_2 = 0$$
$$R_2 = 822 \text{ lb.}$$
$$\Sigma M_{R^2} = 260 \times 7{,}500 - 292 R_1 = 0$$
$$R_1 = 6{,}678 \text{ lb.}$$

Design load per wheel,

$$\frac{(6{,}678 - 375)}{2} \times 5.35 = 16{,}861 \text{ lb.}$$

Design component parallel to thrust line,

$$-16{,}861 \sin 10° = -16{,}861 \times 0.1736 = -2{,}927 \text{ lb.}$$

Design component perpendicular to thrust line,

$$+16{,}861 \times \cos 10° = +16{,}861 \times 0.9848 = +16{,}605 \text{ lb.}$$

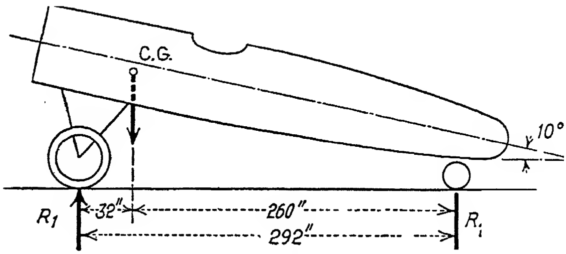

FIG. 206.

To obtain the resultant reactions it is only necessary to add algebraically the products of the reactions due to unit loads and the V and D components of the design load acting at the axle,

$$S_1 = +16{,}605 \times (0) - 2{,}927 \times (-0.0412) = -121 \text{ lb.}$$
$$S_2 = +16{,}605 \times (+0.0702) - 2{,}927 \times (-0.0625) + 1{,}166 + 183 = +1{,}349 \text{ lb.}$$
$$S_3 = +16{,}605 \times (0) - 2{,}927 \times (-0.0253) = +74 \text{ lb.}$$
$$S_4 = +16{,}605 \times (+0.0234) - 2{,}927 \times (+0.0825) + 389 - 183 = +206 \text{ lb.}$$
$$P_A = +16{,}605 \times (-1.293) - 2{,}927 \times (0) = -21{,}470 \text{ lb.}$$
$$P_B = +16{,}605 \times (+0.1066) - 2{,}927 \times (+1.1665) + 1{,}770 - 3{,}414 = -1{,}644 \text{ lb.}$$
$$P_C = +16{,}605 \times (+0.2634) - 2{,}927 \times (-1.3819) + 4{,}374 + 4{,}045 = +8{,}419 \text{ lb.}$$

Bending in member B,

$$M_B = \sqrt{S_1^2 + S_2^2} \times L_B = \sqrt{121^2 + 1{,}349^2} \times 65.76 = 89{,}039 \text{ in lb.}$$

Bending in member C,

$$M_C = \sqrt{S_3^2 - S_4^2} \times L_C = \sqrt{74^2 + 206^2} \times 78.26 = 17{,}139 \text{ in. lb.}$$

311. Fuselage Reactions.—For the fuselage analysis it is necessary to know the V, D, and H components of the resultant

landing-gear reaction from each member. The reactions on the fuselage are of opposite sign to those on the landing gear.

Member B,

$$\sum V = -P_B \times \frac{V}{L_B} - S_1 \times \frac{V}{S_1} - S_2 \times \frac{V}{S_2}$$
$$+1{,}644 \times 0.6083 + 121 \times 0.1425 - 1{,}349 \times 0.7809$$
$$+1{,}000 + 17 - 1{,}053 = -36 \text{ lb.}$$

$$\sum D = +P_B \times \frac{D}{L_B} - S_1 \times \frac{D}{S_1}$$
$$-1{,}644 \times 0.2281 + 121 \times 0.9737$$
$$-375 + 118 = +257 \text{ lb.}$$

$$\sum H = -P_B \times \frac{H}{L_B} - S_1 \times \frac{H}{S_1} + S_2 \times \frac{H}{S_2}$$
$$+1{,}644 \times 0.7603 + 121 \times 0.1781 + 1{,}349 \times 0.6247$$
$$+1{,}250 + 22 + 843 = +2{,}115 \text{ lb.}$$

Member C,

$$\sum V = -P_C \times \frac{V}{L_C} - S_3 \times \frac{V}{S_3} - S_4 \times \frac{V}{S_4}$$
$$-8{,}419 \times 0.5111 - 74 \times 0.3592 - 206 \times 0.7809$$
$$-4{,}303 - 27 - 161 = -4{,}491 \text{ lb.}$$

$$\sum D = -P_C \times \frac{D}{L_C} + S_3 \times \frac{D}{S_3}$$
$$-8{,}419 \times 0.575 + 74 \times 0.8182$$
$$= -4{,}841 + 61 = -4{,}780 \text{ lb.}$$

$$\sum H = -P_C \times \frac{H}{L_C} - S_3 \times \frac{H}{S_3} + S_4 \times \frac{H}{S_4}$$
$$-8{,}419 \times 0.6389 - 74 \times 0.4490 + 206 \times 0.8247$$
$$-5{,}379 - 33 + 129 = -5{,}283 \text{ lb.}$$

Member A,

$$\sum V = +P_A \times \frac{V}{L_A} = +21{,}470 \times 0.9843 = +21{,}133 \text{ lb.}$$

$$\sum D = +P_A \times \frac{D}{L_A} = +21{,}470 \times 0.0984 = +2{,}113 \text{ lb.}$$

$$\sum H = +P_A \times \frac{H}{L_A} = +21{,}470 \times 0.1476 = +3{,}169 \text{ lb.}$$

Check,

$$\Sigma V = +21{,}133 - 36 - 4{,}491 = +16{,}606 \text{ lb.}$$
$$\Sigma D = +2{,}113 - 257 - 4{,}780 = -2{,}924 \text{ lb.}$$
$$\Sigma H = +3{,}169 + 2{,}115 - 5{,}283 = -1 \text{ lb.}$$

312. Equations of Equilibrium, Case III.—Here we have all nine of the balancing reactions shown in Fig. 201. Since there are only the six equations of static equilibrium, it is necessary to have three additional equations in order to solve the problem. The method of least work will be resorted to in order to obtain these equations.

In this case it will be more convenient first to solve for the torsion and bending moments in the members at the apex of the tripod due to applied couples and later to find the values of shear and direct reactions. To accomplish this more easily, new reference axes are taken. Referring to Fig. 204, the X-axis is the same as the D-axis in Fig. 201; the Y-axis is perpendicular to the X-axis and lies in the plane of members B and C; and the Z-axis is perpendicular to the other two axes. In this step we are not concerned with member A since it cannot resist torsion or bending.

The new direction cosines are given in Table XXX.

TABLE XXX.—DIRECTION COSINES OF MEMBERS

Member	Y	X	Z	L	$\frac{Y}{L}$	$\frac{X}{L}$	$\frac{Z}{L}$
B	64.03	15	0	65.76	0.9737	0.2281	0
C	64.03	45	0	78.26	0.8182	0.5750	0

C_1, C_2, and C_3 are applied couples parallel to the three reference axes as shown in Figs. 204 and 205. M_B and M_C are the bending moments in members B and C respectively in planes perpendicular to the plane of the members, while M'_B and M'_C are bending moments in members B and C respectively in the plane of the members. As before, T_B and T_C represent the torsion. We will first deal with C_1 and C_2, and later with C_3.

Figure 205 shows the applied couples and balancing torsions and moments acting at the apex O. The moments are represented by vectors perpendicular to the members and the torsions by vectors coincident with the members as in Case I and II. All vectors are assumed positive.

Writing the equations of equilibrium,

$$\sum M_Y = +M_B \times \frac{V}{L_B} + M_C \times \frac{V}{L_C} + T_B \times \frac{X}{L_B} + T_C \times \frac{X}{L_C} + C_2 = 0$$

$$+M_B \times 0.2281 + M_C \times 0.575 + T_B \times 0.9737 + T_C \times 0.8182 + C_2 = 0 \quad (615)$$

$$\sum M_X = +M_B \times \frac{X}{L_B} - M_C \times \frac{X}{L_C} - T_B \times \frac{Y}{L_B} + T_C \times \frac{V}{L_C} + C_1 = 0$$

$$+ M_B \times 0.9737 - M_C \times 0.8182 - T_B \times 0.2281 + T_C \times 0.5750 + C_1 = 0 \quad (616)$$

313. Work or Energy Equations, Case III.—The work due to bending in a member of uniform section, simply supported with an applied moment at one end, is given by the expression

$$u_b = \frac{M^2L}{6EI}$$

The work due to torsion in a round tube is given by the expression

$$u_t = \frac{T^2L}{4E_SI}$$

where

E = modulus of elasticity in tension or compression.

E_S = modulus of elasticity in shear.

Writing the equation of the summation of the work in the structure (the work due to tension and compression is neglected because it would affect the results but slightly and would greatly increase the labor),

$$\sum W = \frac{M_B^2L_B}{6EI_B} + \frac{M_C^2L_C}{6EI_C} + \frac{T_B^2L_B}{4E_SI_B} + \frac{T_C^2L_C}{4E_SI_B} \quad (617)$$

For steel, $E_S = 0.4E$

Let $K = \frac{L_C}{L_B} \times \frac{L_B}{I_C}$

Substituting these values in (617),

$$W = \frac{L_B}{EI_B}\left[\frac{M_B^2}{6} + K\frac{M_C^2}{6} + \frac{T_B^2}{1.6} + K\frac{T_C^2}{1.6}\right] \quad (618)$$

Here as in Case II it is advisable to assume a value of K for the first approximation. To illustrate the problem, however, we

will assume that the first approximation has already been made and will next try the same tube sizes as given in Case II.

$$I_B = 0.6514 \text{ in.}^4 \qquad I_C = 0.5663 \text{ in.}^4$$

$$K = \frac{78.26}{65.76} \times \frac{0.6514}{0.5663} = 1.369 \tag{619}$$

Substituting (619) in (618),

$$W = \frac{L_B}{EI_B}[0.1667M_B^2 - 0.2282M_C^2 - 0.625T_B^2 - 0.8556T_C^2] \tag{620}$$

314. Solution of Equations, Case III.—There are now two equations of equilibrium and, by taking partial derivatives of equation (620), two additional equations may be obtained, thus giving a solution for the four unknowns.

From equations (615) and (616) the values of T_B and T_C in terms of M_B, M_C, C_1 and C_2 are obtained,

$$T_B = +M_B \times 0.8914 - 1.3396\, M_C + M_1 \times 1.0959 - C_2 \times 0.7703 \tag{621}$$

$$T_C = -M_B \times 1.3396 + 0.8915\, M_C - M_2 \times 0.3056 - C_1 \times 1.3043 \tag{622}$$

From equation (620), applying the theory of least work,

$$\frac{\partial W}{\partial M_B} = \frac{L_B}{EI_B} \left[2 \times 0.1667M_B + 2 \times 0.625T_B\frac{\partial T_B}{\partial M_B} + 2 \times 0.8556T_C\frac{\partial T_C}{\partial M_B}\right] = 0$$

$$0.1667M_B + 0.625T_B\frac{\partial T_B}{\partial M_B} + 0.8556T_C\frac{\partial T_C}{\partial M_B} = 0 \tag{623}$$

$$\frac{\partial W}{\partial M_C} = \frac{L_B}{EI_B} \times \left[2 \times 0.2282M_C + 2 \times 0.625T_B\frac{\partial T_B}{\partial M_C} + 2 \times 0.8556T_C \times \frac{\partial T_C}{\partial M_C}\right] = 0 \tag{624}$$

$$0.2282M_C + 0.625T_B\frac{\partial T_B}{\partial M_C} + 0.8556T_C\frac{\partial T_C}{\partial M_C} = 0 \tag{625}$$

From equation (621),

$$\frac{\partial T_B}{\partial M_C} \quad -1.3396, \qquad \frac{\partial T_B}{\partial M_1} = +0.8914 \tag{626}$$

From equation (622),

$$\frac{\partial T_C}{\partial M_B} = -1.3396, \qquad \frac{\partial T_C}{\partial M_C} = +0.8915 \tag{627}$$

Substituting (621), (622), and (626) in (623),

$$2.1987M_B - 1.7681M_C + 2.1055C_1 - 0.0789C_2 = 0 \tag{628}$$

Substituting (621), (622), and (627) in (625),

$$-1.7681M_B + 2.0291M_C - 1.9124C_1 - 0.4118C_2 = 0 \tag{629}$$

Solving (628) and (629),

$$M_B = -0.6676C_1 - 0.4247C_2 \tag{630}$$

$$M_C = +0.3606C_1 - 0.5728C_2 \tag{631}$$

Substituting (630) and (631) in (621),

$$T_B = +0.0177C_1 - 0.3816C_2 \tag{632}$$

Substituting (630) and (631) in (622),

$$T_C = -0.0885C_1 - 0.2427C_2 \tag{633}$$

The couple C_3 will produce only bending in members B and C in a plane perpendicular to the members and the division of moment will be the same as in Case II.

$$\frac{M'_B}{M'_C} = \frac{I_B}{I_C} \times \frac{L_C}{L_B} = \frac{0.6514}{0.5663} \times \frac{78.26}{65.76} = 1.3689$$

$$M'_B = +1.3689M'_C \tag{634}$$

$$+C'_3 = +M'_B + M'_C = 0 \tag{635}$$

Substituting (634) in (635),

$$M'_C = -0.4221C_3 \tag{636}$$

Substituting (636) in (635),

$$M'_B = -0.5779C_3 \tag{637}$$

Since we now have the distribution of torsion and moments in members A and B due to applied couples acting in three planes, the next step is to solve for the balancing reactions, shown in Fig. 201, which are due to unit loads.

315. Case III, Solution for Unit Vertical Load.—Refer to Figs. 201 and 202.

$C_2 = 0, \qquad \text{and} \qquad C_3 = 0.$

$C_1 = -6 \times 1 = -6.$

$M_B = -0.6676C_1 = -0.6676 \times (-6) = +4.0056$ in.-lb.

$T_B = +0.0177C_1 = +0.0177 \times (-6) = -0.1062$ in.-lb.

$$M_C = +0.3606C_1 = -0.3606 \times (-6) = -2.1636 \text{ in.-lb.}$$
$$T_C = -0.0885C_1 = -0.0885 \times (-6) = +0.5310 \text{ in.-lb.}$$
$$S_2 = \frac{M_B}{L_B} = \frac{4.0056}{65.76} = +0.0609 \text{ lb.}$$
$$S_4 = \frac{M_C}{L_C} = \frac{2.1636}{78.26} = +0.0276 \text{ lb.}$$
$$S_1 = 0$$
$$S_3 = 0$$

It should be noted here that the shears are given their proper signs by inspection as a positive bending moment does not necessarily give a positive shear.

Taking moments about an axis passing through the upper ends of members B and C,

$$\sum M_{BC} = -1 \times (B_H + 6) - T_B \times \frac{D}{T_B} + T_C \times \frac{D}{T_C} - P_A \times \frac{V}{L_A} \times B_H + P_A \times \frac{H}{L_A} \times B_V \quad 0$$

$$1 \times (50 + 6) + 0.1062 \times 0.2281 + 0.5310 \times 0.575 - P_A \times 0.9843 \times 50 + P_A \times 0.1476 \times 40 = 0$$
$$-56 + 0.0242 + 0.3053 - 49.215P_A - 5.904P_A = 0$$
$$P_A = -\frac{55.67}{43.31} = -1.285 \text{ lb.}$$

Taking moments about an axis through the upper end of member B and parallel to the H-axis,

$$\sum M_{BH} = -1 \times B_D - P_A \times \frac{V}{L_A} \times B_D - P_A \times \frac{D}{L_A} \times B_V + T_B \times \frac{H}{T_B} + T_C \times \frac{H}{T_C} - S_4 \times \frac{V}{S_4} \times (B_D + C_D) - P_C \times \frac{V}{L_C} \times (B_D + C_D) = 0$$

$$-1 \times 15 + 1.285 \times 0.9843 \times 15 + 1.285 \times 0.0984 \times 40 - 0.1062 \times 0.7603 + 0.5310 \times 0.6389 - 0.0276 \times 0.7809 \times (15 + 45) - 6 \times 0.5111 \times (15 + 45) = 0$$
$$-15 + 18.972 + 5.058 - 0.0807 + 0.339 - 1.293 - 30.67P_C = 0$$
$$P_C = +\frac{7.9953}{36.67} = +0.2607 \text{ lb.}$$

Taking moments about an axis through the upper end of member C and parallel to the H-axis,

$$\sum M_{CH} = +1 \times C_D + P_A \times \frac{V}{L_A} \times C_D - P_A \times \frac{D}{L_A} \times C_V + T_B \times \frac{H}{T_B} + T_C \times \frac{H}{T_C} + S_2 \times \frac{V}{S_2} \times (B_D + C_D) + P_B \times \frac{V}{L_B} \times (B_D + C_D) = 0$$

$$+1 \times 45 - 1.285 \times 0.9843 \times 45 + 1.285 \times 0.0984 \times 40 - 0.1062 \times 0.7603 + 0.5310 \times 0.6389 + 0.0609 \times 0.7809 \times (15 + 45) + P_B \times 0.6083 \times (15 + 45) = 0$$

$$+45 - 56.917 + 5.058 - 0.0807 + 0.339 + 2.853 + 36.56P_B = 0$$

$$P_B = +\frac{3.7477}{36.5} = +0.1027 \text{ lb.}$$

316. Case III, Solution for Unit Drag Load.—Refer to Figs. 202, 203 and 205.

$$C_1 = +3.9 \times 1 = +3.9$$
$$C_2 = 0$$
$$C_3 = +4.562 \times 1 = +4.562$$
$$M_B = -0.4247 \times 3.9 = -1.6563 \text{ in. lb.}$$
$$T_B = -0.3816 \times 3.9 = -1.4882 \text{ in lb.}$$
$$M_C = -0.5728 \times 3.9 = -2.2339 \text{ in lb.}$$
$$T_C = -0.2474 \times 3.9 = 0.9649 \text{ in lb.}$$
$$M'_B = -0.5779 \times 4.562 = -2.6362 \text{ in lb.}$$
$$M'_C = -0.4221 \times 4.562 = -1.9258 \text{ in lb.}$$
$$S_1 = \frac{M'_B}{L_B} = \frac{2.6362}{65.76} = +0.0401 \text{ lb.}$$
$$S_2 = \frac{M_B}{L_B} = \frac{1.6563}{65.76} = -0.0252 \text{ lb.}$$
$$S_3 = \frac{M'_C}{L_B} = \frac{1.9258}{78.26} = -0.0246 \text{ lb.}$$
$$S_4 = \frac{M_C}{L_C} = \frac{2.2339}{78.26} = +0.0285 \text{ lb.}$$

Taking moments about an axis passing through the upper ends of members B and C,

$$\sum M_{BC} = -T_B \times \frac{D}{T_B} + T_C \times \frac{D}{T_C} - P_A \times \frac{V}{L_A} \times B_H + P_A \times \frac{H}{L_A} \times B_V = 0$$

$$+1.4882 \times 0.2281 - 0.9649 \times 0.575 - P_A \times 0.9843 \times 50 + P_A \times 0.1476 \times 40 = 0$$

$$+0.3395 - 0.5548 - P_A \times 49.125 + P_A \times 5.904 = 0$$

$$P_A = -\frac{0.2153}{43.311} = -0.005 \text{ lb.}$$

Taking moments about an axis through the upper end of member B and parallel to the H-axis,

$$\sum M_{BH} = +T_B \times \frac{H}{T_B} + T_C \times \frac{H}{T_C} - 1 \times B_V - P_A \times \frac{V}{L_A} \times B_D - P_A \times \frac{D}{L_A} \times B_V - S_4 \times \frac{V}{S_4}(B_D + C_D) - S_3 \times \frac{V}{S_3} \times (B_D + C_D) - P_C \times \frac{V}{L_C} \times (B_D + C_D) = 0$$

$$-1.4882 \times 0.7603 - 0.9649 \times 0.6389 - 1 \times 40 + 0.005 \times 0.9843 \times 15 + 0.005 \times 0.0984 \times 40 - 0.0285 \times 7809 \times 60 + 0.0246 \times 0.3592 \times 60 - P_C \times 0.5111 \times 60 = 0$$

$$-1.1315 - 0.6165 - 40 + 0.0738 + 0.0197 - 1.3353 + 0.5302 - P_C \times 30.67 = 0$$

$$P_C = -\frac{42.46}{30.67} = -1.3844 \text{ lb.}$$

Taking moments about an axis through the upper end of member C and parallel to the H-axis,

$$\sum M_{CH} = +T_B \times \frac{H}{T_B} + T_C \times \frac{H}{T_C} - 1 \times B_V + P_A \times \frac{V}{L_A} \times C_D - P_A \times \frac{D}{L_A} \times C_V + S_2 \times \frac{V}{S_2} \times (B_D + C_D) + S_1 \times \frac{V}{S_1} \times (B_D + C_D) + P_B \times \frac{V}{L_B} \times (B_D + C_D)$$

$$-1.4882 \times 0.7603 - 0.9649 \times 0.6389 - 1 \times 40 - 0.005 \times 0.9843 \times 45 + 0.005 \times 0.0984 \times 40 - 0.0252 \times 0.7809 \times 60 + 0.0401 \times 0.1425 \times 60 + P_B \times 0.6083 \times 60 = 0$$

$$-1.1315 - 0.6165 - 40 - 0.2215 + 0.0197 - 1.1807 + 0.3429 + 36.5 \times P_B = 0$$

$$P_B = +\frac{42.7876}{36.5} = +1.1723 \text{ lb.}$$

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